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Supersymmetric Mechanics - Vol. 3

Attractors and Black Holes in Supersymmetric Gravity

Springer

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Preface

This is the third volume in a series of books on the general topics of Supersymmetric Mechanics, with the first and second volumes being published as Lecture Notes in Physics Vol. 698, Supersymmetric Mechanics – Vol. 1: Supersymmetry, Noncommutativity and Matrix Models (ISBN: 3-540-33313-4), and Lecture Notes in Physics Vol. 701, Supersymmetric Mechanics – Vol. 2: The Attractor Mechanism and Space Time Singularities (ISBN: 3-540-34156-0).

The aim of this ongoing collection is to provide a reference corpus of suitable, introductory material to the field, by gathering the significantly expanded and edited versions of all tutorial lectures, given over the years at the well-established annual INFN-Laboratori Nazionali di Frascati Winter School on the Attractor Mechanism, directed by myself.

The present set of notes results again from the participation and dedication of prestigious lecturers, such as Iosif Bena, Sergio Ferrara, Renata Kallosh, Per Kraus, Finn Larsen, and Boris Pioline. As usual, the lectures were subsequently carefully edited and reworked, taking into account the extensive follow-up discussions. The present volume emphasizes topics of great recent interest, namely general concepts of attractors in supersymmetric gravity and black holes.

A two-parameter family of spherically symmetric, static, asymptotically flat, electrically charged singular metrics in $d = 4$ is described by the so-called Reissner-Nördstrom solution. It may be rigorously shown that the spherically symmetric solution of $N = 2,\ d = 4$ Maxwell-Einstein supergravity represented by an extremal Reissner-Nördstrom black hole preserves one-half of the supersymmetry isometries out of the eight related to the asymptotical limit given by the $N = 2,\ d = 4$ Minkowski background.

When approaching the event horizon of the black hole, one gets a restoration of the previously lost four additional supersymmetries, hence reobtaining a maximally symmetric $N = 2$ metric background, namely the 4-d Bertotti-Robinson $AdS \times S^2$ black hole metric.

In the earlier book Supersymmetric Mechanics – Vol. 2, a general dynamical principle was considered, namely the "attractor mechanism", which governs the dynamics inside the moduli space, with supersymmetry being related to dynamical systems with fixed points describing the corresponding equilibrium state and the stability properties. If this mechanism holds, in approaching some fixed values, which depend solely upon the electric and magnetic charges of the theory, the orbits of the dynamical evolution lose all memory of their initial conditions, and yet the overall dynamics remains fully deterministic. Historically, the first attractor example in supersymmetric systems emerged from the consideration of extreme black holes in $N = 2,\ d = 4,5$ Maxwell-Einstein supergravities coupled with matter multiplets. In the present volume, some of the founders of the research in this field, interacting among themselves, as well as with younger collaborators, yield a pedagogical introduction to the subject.

In his lectures, Iosif Bena (co-authored by Nick Warner) gives an introduction to the construction and analysis of three-charge configurations in string theory and supergravity and describes the corresponding implications for the physics of black holes in string theory. Sergio Ferrara (co-authored by Mike Duff) reviews some recently established connections between the mathematics of black hole entropy in string theory and that of multipartite entanglement in quantum information theory, a topic that could be of great interest also for experimental testing and perhaps even for potential applications. The lectures by Renata Kallosh (co-authored by Stefano Bellucci, Sergio Ferrara, and Alessio Marrani) provides a pedagogical, introductory review of the Attractor Mechanism (at work in two different 4-dimensional frameworks: extremal black holes in $N = 2$ supergravity and $N = 1$ flux compactifications. AdS_3 black holes and their connection to two-dimensional conformal field theories via the AdS/CFT correspondence are the subject of the lectures by Per Kraus, including background material on gravity in AdS_3, in the context of the holographic renormalization. Also Finn Larsen in his lectures yields a pedagogical introduction to the attractor mechanism, in particular in five dimensions, concentrating chiefly on supersymmetry-preserving black holes in five dimensions, both with and without spherical symmetry, being motivated essentially by the consideration of black rings, as well as rotating black holes. Pioline in his contribution “Black Holes, Topological Strings and Quantum Attractors” reviews recent developments on the relation between the macroscopic entropy of four-dimensional BPS black holes and the microscopic counting of states.

I wish to thank all lecturers and participants of the School for contributing to create an almost magical atmosphere to progress in the learning and the further researching in this absolutely fascinating topic. I wish to thank most warmly Mrs. Silvia Colasanti for her generous efforts in the secretarial work and in various organizational aspects. My gratitude goes to INFN and in particular to Mario Calvetti for supporting the School. In welcoming our brand new daughter Erica, my thoughts go to my wife Gloria and our beloved Costanza, Eleonora, and Annalisa for providing me everyday joy, without which I could never have accomplished this effort.

Frascati, December 2007 *Stefano Bellucci*

Contents

Black Holes, Black Rings, and their Microstates

Iosif Bena and Nicholas P. Warner

Abstract In this review article, we describe some of the recent progress towards the construction and analysis of three-charge configurations in string theory and supergravity. We begin by describing the Born-Infeld construction of three-charge supertubes with two dipole charges and then discuss the general method of constructing three-charge solutions in five dimensions. We explain in detail the use of these methods to construct black rings, black holes, as well as smooth microstate geometries with black hole and black ring charges, but with no horizon. We present arguments that many of these microstate geometries are dual to boundary states that belong to the same sector of the D1-D5-P CFT as the typical states. We end with an extended discussion of the implications of this work for the physics of black holes in string theory.

1 Introduction

Black holes are very interesting objects, whose physics brings quantum mechanics and general relativity into sharp contrast. Perhaps the best known, and sharpest, example of such contrast is Hawking's information paradox [1, 2]. This has provided a very valuable guide and testing ground in formulating a quantum theory of gravity. Indeed, it is one of the relatively few issues that we know *must* be explained by a viable theory of quantum gravity.

String theory is a quantum theory of gravity and has had several astounding successes in describing properties of black holes. In particular, Strominger and Vafa have shown [3] that one can count microscopic configurations of branes and strings at zero gravitational coupling and exactly match their statistical entropy to

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the Bekenstein-Hawking entropy of the corresponding black hole at large effective coupling.

Another way to understand the Strominger-Vafa entropy matching is via the *AdS*-CFT correspondence[1] [4, 5, 6]. One can make a black hole in string theory by putting together D5 branes and D1 branes and turning on momentum along the direction of the D1s. If one takes a near horizon limit of this system, one finds a bulk that is asymptotic to $AdS_3 \times S^3 \times T^4$, and which contains a BPS black hole. The dual boundary theory is the two-dimensional conformal field theory that lives on the intersection of the D1 branes and the D5 branes and is known as the D1-D5-P CFT. If one counts the states with momentum N_p and R-charge J in this conformal field theory, one obtains the entropy

$$S = 2\pi\sqrt{N_1 N_5 N_p - J^2}, \tag{1}$$

which precisely matches the entropy of the dual black hole [7, 8] in the bulk.

A very important question, with deep implications for the physics of black holes, is: "What is the fate of these microscopic brane configurations as the effective coupling becomes large?" Alternatively, the question can be rephrased in *AdS*-CFT language as: "What is the gravity dual of individual microstates of the D1-D5-P CFT?" More physically, "What do the black-hole microstates look like in a background that a relativist would recognize as a black hole?"

1.1 Two-Charge Systems

These questions have been addressed for the simpler D1-D5 system[2] by Mathur, Lunin, Maldacena, Maoz, and others [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]; see [25, 26] for earlier work in this direction, and [27, 28] for a review of that work. They found that the states of that CFT can be mapped into two-charge supergravity solutions that are asymptotically $AdS_3 \times S^3 \times T^4$ and have no singularity. These supergravity solutions are determined by specifying an arbitrary closed curve in the space transverse to the D1 and D5 branes and have a dipole moment corresponding to a Kaluza-Klein monopole (KKM) wrapped on that curve.[3] Counting these configurations [9, 29, 30, 31, 32] has shown that the entropy of the CFT is reproduced by the entropy coming from the arbitrariness of the shape of the closed curve.

While the existence of such a large number of two-charge supergravity solutions might look puzzling – again, these BPS solutions are specified by arbitrary

[1] Historically the AdS-CFT correspondence was found later.

[2] Throughout these lectures we will refer to the D1-D5 system and its U-duals as the two-charge system, and to the D1-D5-P system and its U-duals as the three-charge system.

[3] A system that has a prescribed set of charges as measured from infinity often must have additional dipole charge distributions. We will discuss this further in Sect. 2, but for the present, one should note the important distinction between asymptotic charges and dipole charges.

functions – there is a simple string-theoretic reason for this. By performing a series of S and T dualities, one can dualize the D1-D5 configurations with KKM dipole charge into configurations that have F1 and D0 charge, and D2-brane dipole moment. Via an analysis of the Born-Infeld action of the D2 brane, these configurations were found by Mateos and Townsend to be supersymmetric, and moreover to preserve the same supersymmetries as the branes whose asymptotic charges they carry (F1 and D0 charge), independent of the shape of the curve that the D2 brane wraps [33, 34, 35]. Hence, they were named "supertubes." Alternatively, one can also dualize the D1-D5 (+ KKM dipole) geometries into F1 string configurations carrying left-moving momentum. Because the string only has transverse modes, the configurations carrying momentum will have a non-trivial shape: Putting the momentum into various harmonics causes the shape to change accordingly. Upon dualizing, the shape of the momentum wave on the F1 string can be mapped into the shape of the supertube [36].

Thus, for two-charge system, we see that the existence of a large number of supergravity solutions could have been anticipated from this earlier work on the microscopic two-charge stringy configurations obtained from supertubes and their duals. In Section 2, we will consider three-charge supertubes and discuss how this anticipated the discovery of some of the corresponding supergravity solutions that are discussed in Section 3.

1.2 Implications for Black-Hole Physics

An intense research programme has been unfolding over the past few years to try to see whether the correspondence between D1-D5 CFT states and smooth bulk solutions also extends to the D1-D5-P system. The crucial difference between the two-charge system and the three-charge system (in five dimensions) is that the latter generically has a macroscopic horizon, whereas the former only has an effective horizon at the Planck or string scale. Indeed, historically, the link between microstate counting and Bekenstein-Hawking entropy (at vanishing string coupling) was first investigated by Sen [37] for the two-charge system. While this work was extremely interesting and suggestive, the result became compelling only when the problem was later solved for the three-charge system by Strominger and Vafa [3]. Similarly, the work on the microstate geometries of two-charge systems is extremely interesting and suggestive, but to be absolutely compelling, it must be extended to the three-charge problem. This would amount to establishing that the boundary D1-D5-P CFT microstates are dual to bulk microstates – configurations that have no horizons or singularities, and which look like a black hole from a large distance, but start differing significantly from the black hole solution at the location of the would-be horizon.

String theory would then indicate that a black hole solution should not be viewed as a fundamental object in quantum gravity but rather as an effective "thermodynamic" description of an ensemble of horizonless configurations with the same

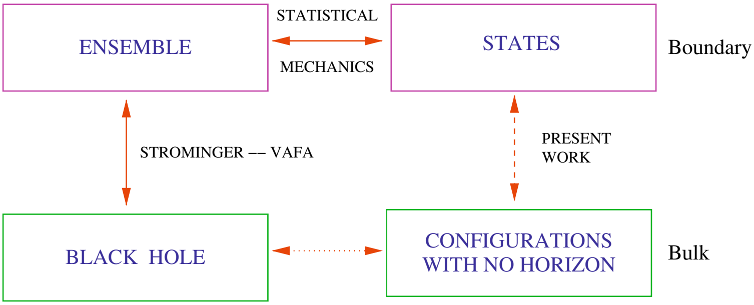

Fig. 1 An illustrative description of Mathur's conjecture. Most of the present research efforts go into improving the dictionary between bulk and boundary microstates (*the dotted arrow*), and into constructing more microstate geometries

macroscopic/asymptotic properties (see Fig. 1). The black hole horizon would be the place where these configurations start differing from each other, and the classical "thermodynamic" description of the physics via the black hole geometry stops making sense.

An analogy that is useful in understanding this proposal is to think about the air in a room. One can use thermodynamics and fluid mechanics to describe the air as a continuous fluid with a certain equation of state. One can also describe the air using statistical mechanics, by finding the typical configurations of molecules in the ensemble, and noticing that the macroscopic features of these configurations are the same as the ones found in the thermodynamic description. For most practical purposes, the thermodynamic description is the one to use; however, this description fails to capture the physics coming from the molecular structure of the air. To address problems like Brownian motion, one should not use the thermodynamic approximation but the statistical description. Similarly, to address questions having to do with physics at the scale of the horizon (like the information paradox) one should not use the thermodynamic approximation given by the black hole solution, but one should use the statistical description given by the microstate configurations.

This dramatic shift in the description of black holes has been most articulately proposed and strongly advocated by Mathur and is thus often referred to as "Mathur's conjecture." In fact, one should be careful and distinguish two variants of this conjecture. The weak variant is that the black hole microstates are horizon-sized stringy configurations that have unitary scattering but cannot be described accurately using the supergravity approximation. These configurations are also sometimes called "fuzzballs." If the weak Mathur conjecture were true then the typical bulk microstates would be configurations where the curvature is Planck scale and hence cannot be described in supergravity. The strong form of Mathur's conjecture, which is better defined and easier to prove or disprove, is that among the *typical* black hole microstates there are smooth solutions that can be described using supergravity.

Of course, the configurations that will be discussed and constructed in these notes are classical geometries with a moduli space. Classically, there is an infinite number of such configurations that need to be quantized before one can call them *micro*states in the strictest sense of the word. In the analogy with the air in a room, these geometries correspond to classical configurations of molecules. Classically, there is an infinite number of such configurations, but one can quantize them and count them to find the entropy of the system.

Whichever version of the conjecture is correct, we are looking for stringy configurations that are very similar to the black hole from far away and start differing from each other at the location of the would-be horizon. Thus black hole microstates should have a size of the same order as the horizon of the corresponding black hole. From the perspective of string theory, this is very a peculiar feature, since most of the objects that one is familiar with become smaller, not larger, as gravity becomes stronger. We will see in these lectures how our black hole microstates manage to achieve this feature.

If the strong form of this conjecture were true then it would not only solve Hawking's information paradox (microstates have no horizon, and scattering is unitary) but also would have important consequences for quantum gravity. It also might allow one to derive 't Hooft's holographic principle from string theory, and might even have experimental consequences. A more detailed discussion about this can be found in Sect. 9.

1.3 Outline

As with the two-charge systems, the first step in finding three-charge solutions that have no horizon and look like a black hole is to try to construct large numbers of microscopic stringy three-charge configurations. This is the subject of Sect. 2, in which we review the construction of three-charge supertubes – string theory objects that have the same charges and supersymmetries as the three-charge black hole [38].

In Sect. 3, we present the construction of three-charge supergravity solutions corresponding to arbitrary superpositions of black holes, black rings, and three-charge supertubes of arbitrary shape. We construct explicitly a solution corresponding to a black hole at the center of a black ring and analyze the properties of this solution. This construction and the material presented in subsequent sections can be read independently of Sect. 2.

Section 4 is a geometric interlude devoted to Gibbons-Hawking metrics and the relationship between five-dimensional black rings and four-dimensional black holes. Section 5 contains the details of how to construct new microstate solutions using an "ambipolar" Gibbons-Hawking space, whose signature alternates from $(+,+,+,+)$ to $(-,-,-,-)$. Even though the sign of the base-space metric can flip, the full eleven-dimensional solutions are smooth.

In Sect. 6, we discuss geometric transitions and the way to obtain smooth horizonless "bubbling" supergravity solutions that have the same type of charges and angular momenta as three-charge black holes and black rings. In Sect. 7, we construct several such solutions, finding, in particular, microstates corresponding to zero-entropy black holes and black rings.

In Sect. 8, we use mergers to construct and analyze “deep microstates,” which correspond to black holes with a classically large horizon area. We find that the depth of these microstates becomes infinite in the classical (large charge) limit and argue that they correspond to CFT states that have one long component string. This is an essential (though not sufficient) feature of the duals of typical black-hole microstates (for reviews of this, see [39, 40]). Thus the “deep microstates” are either typical microstates themselves, or at least lie in the same sector of the CFT as the typical microstates.

Finally, Sect. 9 contains conclusions and an extensive discussion of the implications of the work presented here on for the physics of black holes in string theory.

Before beginning, we should emphasize that the work that we present is part of a larger effort to study black holes and their microstates in string theory. Many groups have worked at obtaining smooth microstate solutions corresponding to five-dimensional and four-dimensional black holes, a few of the relevant references include [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. Other groups focus on improving the dictionary between bulk microstates and their boundary counterparts, both in the two-charge and in the three-charge systems [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 56]. Other groups focus on small black holes[4] and study their properties using the attractor mechanism [58] or relating them to topological strings via the OSV conjecture [59]. Reviews of this can be found in [60, 61], and a limited sample of work that is related to the exploration presented here can be found in [62, 63, 64, 65, 66, 145, 146, 147, 148, 149].

2 Three-Charge Microscopic Configurations

Our purpose here is to follow the historical path taken with the two-charge system and try to construct three-charge brane configurations using the Born-Infeld (BI) action. We are thus considering the intrinsic action of a brane, and we will not consider the back-reaction of the brane on the geometry. The complete supergravity solutions will be considered later.

There are several ideas in the study of D-branes that will be important here. First, one of the easiest ways to create a system with multiple, different brane charges is to start with a higher-dimensional brane and then turn on electromagnetic fields on that brane so as to induce lower-dimensional branes that are “dissolved” in the original brane. We will use this technique to get systems with D0-D2-D4-D6 charges below.

In constructing multi-charge solutions, one should also remember that the equations of motion are generically non-linear. For example, in supergravity the Maxwell

[4] These black holes do not have a macroscopic horizon, but one can calculate their horizon area using higher order corrections [57]. This area agrees with both the CFT calculation of the entropy and also agrees (up to a numerical factor) with the counting of two-charge microstates. Hence, one could argue (with a caveat having to do with the fact that small black holes in IIA string theory on T^4 receive no corrections) that small black holes, which from the point of view of string theory are in the same category as the big black holes, are, in fact, superpositions of horizonless microstates.

action can involve Chern-Simons terms, or the natural field strength may involve wedge products of lower degree forms. Similarly, in the BI action there is a highly non-trivial interweaving of the Maxwell fields and hence of the brane charges. In practice, this often means that one cannot simply lay down independent charges: Combinations of fields sourced by various charges may themselves source other fields and thus create a distribution of new charges. In this process, it is important to keep track of asymptotic charges, which can be measured by the leading fall-off behavior at infinity, and "dipole" distributions that contribute no net charge when measured at infinity. When one discusses an N-charge system, one means a system with N commuting asymptotic charges, as measured at infinity. For microstate configurations, one often finds that the systems that have certain charges will also have fields sourced by other dipole charges. More precisely, in discussing the BI action of supertubes we will typically find that a given pair of asymptotic charges, A and B, comes naturally with a third set of dipole charges, C. We will therefore denote this configuration by $A-B \to C$.

2.1 Three-Charge Supertubes

The original two-charge supertube [33] carried two independent asymptotic charges, D0 and F1, as well as a D2-brane dipole moment; thus we denote it as a F1-D0 $\to$ D2 supertube. It is perhaps most natural to try to generalize this object by combining it with another set of branes to provide the third charge.[5] Supersymmetry requires that this new set be D4 branes. To be more precise, supertubes have the same sypersymmetries as the branes whose asymptotic charges they carry, and so one can naturally try to put together F1-D0 $\to$ D2 supertubes, F1-D4 $\to$ D6 supertubes, and D0-D4 $\to$ NS5 supertubes obtain a supersymmetric configuration that has three asymptotic charges: D0, D4, and F1, and three dipole distributions, coming from D6, NS5, and D2 branes wrapping closed curves. Of course, the intuition coming from putting two-charge supertubes together, though providing useful guidance, will not be able to indicate anything about the size or other properties of the resulting three-charge configuration.

Exercise 1. *Show that the supertube with D2 dipole charge and F1 and D0 charges can be dualized into an F1-D4 $\to$ D6 supertube, and into a D0-D4 $\to$ NS5 supertube.*

To investigate objects with the foregoing charges and dipole charges one has to use the theory on *one* of the sets of branes and then describe all the other branes as objects in this theory. One route is to consider tubular D6-branes[6] and attempt

[5] One might also have tried to generalize the F1-P dual of this system by adding a third type of charge. Unfortunately, preserving the supersymmetry requires this third charge to be that of NS5 branes and, because of the dilaton throat of these objects, an analysis of the F1-P system similar to the two-charge one [9] cannot be done.

[6] Tubular means it will only have a dipole charge just like any loop of current in electromagnetism.

to turn on world-volume fluxes to induce D4, D0, and F1 charges. As we will see, such a configuration also has a D2 dipole moment. An alternative route is to use the D4 brane non-Abelian Born-Infeld action. Both routes were pursued in [38], leading to identical results. Nevertheless, for simplicity we will only present the first approach here.

One of the difficulties in describing three-charge supertubes in this way is the fact that the Born-Infeld action and its non-Abelian generalization cannot be used to describe NS5 brane dipole moments. This is essentially because the NS5 brane is a non-perturbative object from the perspective of the Born-Infeld action [67]. Thus, our analysis of three charge supertubes is limited to supertubes that only have D2 and D6 dipole charge. Of course, one can dualize these to supertubes with NS5 and D6 dipole charges or to supertubes with NS5 and D2. Nevertheless, using the action of a single brane, it is not possible to describe supertubes that have three charges and three dipole charges. For that, we will have to wait until Sect. 3, where we will construct the full supergravity solution corresponding to these objects.

2.2 The Born-Infeld Construction

We start with a single tubular D6-brane and attempt to turn on worldvolume fluxes so that we describe a BPS configuration carrying D4, D0, and F1 charges. We will see that this also necessarily leads to the presence of D2-brane charges, but we will subsequently introduce a second D6-brane to cancel this.

The D6-brane is described by the Born-Infeld action

$$S = -T_6 \int d^7\xi \sqrt{-\det(g_{ab} + \mathscr{F}_{ab})}, \tag{2}$$

where g_{ab} is the induced worldvolume metric, $\mathscr{F}_{ab} = 2\pi F_{ab}$, T_6 is the D6-brane tension, and we have set $\alpha' = 1$. The D6 brane also couples to the background RR fields through the Chern-Simons action:

$$S_{CS} = T_6 \int \exp\left(\mathscr{F} + B\right) \wedge \sum_q C^{(q)}. \tag{3}$$

By varying this with respect to the $C^{(q)}$, one obtains the D4-brane, D2-brane, and D0-brane charge densities:

$$Q_4 = 2\pi\, T_6\, \mathscr{F} \tag{4}$$

$$Q_2 = 2\pi\, T_6\, \left(\frac{1}{2}\, \mathscr{F} \wedge \mathscr{F}\right) \tag{5}$$

$$Q_0 = 2\pi\, T_6\, \left(\frac{1}{3!}\, \mathscr{F} \wedge \mathscr{F} \wedge \mathscr{F}\right). \tag{6}$$

To obtain the quantized Dp-brane charges, one takes the volume p-form on any compact, p-dimensional spatial region, R, and wedges this volume form with Q_p

and integrates over the spatial section of the D6 brane. The result is then the Dp-brane charge in the region R.

The F1 charge density can be obtained by varying the action with respect to the time-space component of NS-NS two form potential, B. Since B appears in the combination $\mathscr{F}+B$, one can differentiate with respect to the gauge field:

$$Q_1 = \frac{\partial \mathscr{L}}{\partial B_{0i}} = \frac{\partial \mathscr{L}}{\partial F_{0i}} = \frac{\partial \mathscr{L}}{\partial \dot{\vec{A}}} = \vec{\pi}, \tag{7}$$

which is proportional to the canonical momentum conjugate to the vector potential, $\vec{A}$.

Our construction will essentially follow that of the original D2-brane supertube [33], except that we include four extra spatial dimensions and corresponding fluxes. We take our D6-brane to have the geometry $\mathbb{R}^{1,1} \times S^1 \times T^4$, and we choose coordinates (x^0, x^1) to span $\mathbb{R}^{1,1}$ and (x^6, x^7, x^8, x^9) to span the T^4. The S^1 will be a circle of of radius r in the (x^2, x^3) plane, and we will let θ be the angular coordinate in this plane. We have also introduced factors of 2π in (4), (5), (6) and (7) to anticipate the fact that for round tubes everything will be independent of θ, and so the integrals over θ will generate these factors of 2π. Thus the D-brane charge densities above are really charge densities in the remaining five dimensions, and the fundamental string charge is a charge density per unit four-dimensional area. Note also that the charges, Q, are the ones that appear in the Hamiltonian and are related to the number of strings or branes by the corresponding tensions. These conventions will be convenient later on.

Since the S^1 is contractible and lies in the non-compact space-time, any D-brane wrapping this circle will not give rise to asymptotic charges and will only be dipolar. In particular, the configuration carries no asymptotic D6-brane charge due to its tubular shape. To induce D0-branes we turn on *constant* values of $\mathscr{F}_{1\theta}$, $\mathscr{F}_{67}$, and $\mathscr{F}_{89}$. Turning on $\mathscr{F}_{1\theta}$ induces a density of D4-branes in the (x^6, x^7, x^8, x^9) plane, and since these D4 branes only wrap the T^4, their charge can be measured asymptotically. The fields $\mathscr{F}_{67}$, and $\mathscr{F}_{89}$ similarly generate dipolar D4-brane charges. To induce F1 charge in the x^1 direction we turn on a constant value of $\mathscr{F}_{01}$. It is also evident from (5) that this configuration carries asymptotic D2-brane charges in the (x^6, x^7) and (x^8, x^9) planes and dipolar D2-brane charge in the (x^1, θ) direction. The asymptotic D2-brane charges will eventually be canceled by introducing a second D6-brane. This will also cancel the dipolar D4-brane and D2-brane charges, and we will then have a system with asymptotic F1, D0, and D4 charges and dipolar D2 and D6 charges.

With these fluxes turned on we find

$$S = -T_6 \int d^7\xi \, \sqrt{\left(1 - \mathscr{F}_{01}^2\right) r^2 + \mathscr{F}_{1\theta}^2} \sqrt{\left(1 + \mathscr{F}_{67}^2\right)\left(1 + \mathscr{F}_{89}^2\right)}, \tag{8}$$

where we use polar coordinates in the (x^2, x^3) plane, and the factors of r^2 come from $g_{\theta\theta}$. By differentiating with respect to $\mathscr{F}_{01}$, we find

$$Q_1 = 2\pi T_6 \frac{\mathscr{F}_{01} r^2}{\sqrt{\left(1-\mathscr{F}_{01}^2\right) r^2 + \mathscr{F}_{1\theta}^2}} \sqrt{\left(1+\mathscr{F}_{67}^2\right)\left(1+\mathscr{F}_{89}^2\right)}. \tag{9}$$

The key point to observe now is that if we choose

$$\mathscr{F}_{01} = 1 \tag{10}$$

then r^2 drops out of the action (8). We will also choose

$$\mathscr{F}_{67} = \mathscr{F}_{89}. \tag{11}$$

We can then obtain the energy from the canonical Hamiltonian:

$$H = \int Q_1 \mathscr{F}_{01} - L \tag{12}$$

$$= \int \left[Q_1 + 2\pi T_6 |\mathscr{F}_{1\theta}| + 2\pi T_6 |\mathscr{F}_{1\theta} \mathscr{F}_{67} \mathscr{F}_{89}| \right] \tag{13}$$

$$= \int \left[Q_1 + Q_4 + Q_0 \right]. \tag{14}$$

The last two integrals are taken over the coordinates $(x^1, x^6, x^7, x^8, x^9)$ of the D6-brane. The radius of the system is determined by inverting (9):

$$r^2 = \frac{Q_1}{2\pi T_6} \frac{\mathscr{F}_{1\theta}}{1+\mathscr{F}_{67}\mathscr{F}_{89}} = \frac{1}{(2\pi T_6)^2} \frac{Q_1 Q_4^2}{Q_0 + Q_4}. \tag{15}$$

If we set $Q_0 = 0$ then (15) reduces (with the obvious relabeling) to the radius formula found for the original D2-brane supertube [33]. From (14), we see that we have saturated the BPS bound, and so our configuration must solve the equations of motion, as can be verified directly.

Exercise 2. *Minimize the Hamiltonian in (12) by varying the radius, r, while keeping the F1, D0, and D4 charges constant. Verify that the configuration with the radius given (15) solves the equations of motion.*

Supersymmetry can also be verified precisely as for the original D2-brane supertube [33]. The presence of the electric field, $\mathscr{F}_{01} = 1$, causes the D6-brane to drop out of the equations determining the tension and the unbroken supersymmetry. Indeed, just like the two-charge system [34], we can consider a D6-brane that wraps an *arbitrary* closed curve in $\mathbb{R}^4$; the only change in (8) and (9) is that r^2 will be replace by the induced metric on the D6 brane, $g_{\theta\theta}$. However, when $\mathscr{F}_{01} = 1$ this does not affect (13) and (14), and therefore the configuration is still BPS. Moreover, if $\mathscr{F}_{1\theta}$ is not constant along the tube, or if $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$ remain equal but depend on θ the BPS bound is still saturated.

Hence, classically, there exists an infinite number of three-charge supertubes with two dipole charges, parameterized by several arbitrary functions of one variable [38]. Four of these functions come from the possible shapes of the supertube, and two functions come from the possibility of varying the D4 and D0 brane densities inside the tube. Anticipating the supergravity results, we expect three-charge, three-dipole charge tubes to be given by *seven arbitrary functions*, four coming from the shape and three from the possible brane densities inside the tube. The procedure of constructing supergravity solutions corresponding to these objects [68, 69] will be discussed in the next section, and will make this "functional freedom" very clear.

As we have already noted, the foregoing configuration also carries non-vanishing D2-brane charge associated with $\mathscr{F}_{1\theta}\mathscr{F}_{67}$ and $\mathscr{F}_{1\theta}\mathscr{F}_{89}$. It also carries dipolar D4-brane charges associated with $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$. To remedy this we can introduce one more D6 brane with flipped signs of $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$ [70]. This simply doubles the D4, D0, and F1 charges, while canceling the asymptotic D2 charge and the dipolar D4-brane charges. More generally, we can introduce k coincident D6-branes, with fluxes described by diagonal $k \times k$ matrices. We again take the matrix-valued field strengths $\mathscr{F}_{01}$ to be equal to the unit matrix, in order to obtain a BPS state. We also set $\mathscr{F}_{67} = \mathscr{F}_{89}$ and take $F_{1\theta}$ to have non-negative diagonal entries to preclude the appearance of $\overline{D4}$-branes. The condition of vanishing D2-brane charge is then

$$\mathrm{Tr}\,\mathscr{F}_{1\theta}\,\mathscr{F}_{67} = \mathrm{Tr}\,\mathscr{F}_{1\theta}\mathscr{F}_{89} = 0. \tag{16}$$

This configuration can also have D4-brane dipole charges, which we may set to zero by choosing

$$\mathrm{Tr}\,\mathscr{F}_{67} = \mathrm{Tr}\,\mathscr{F}_{89} = 0. \tag{17}$$

Finally, the F1 charge is described by taking Q_1 to be an arbitrary diagonal matrix with non-negative entries.[7] This results in a BPS configuration of k D6-branes wrapping curves of arbitrary shape. If the curves are circular, the radius formula is now given by (15) but with the entries replaced by the corresponding matrices. Of course, for our purposes we are interested in situations when we can use the Born-Infeld action of the D6 branes to describe the dynamics of our objects. Since the BI action does not take into account interactions between separated strands of branes, we will henceforth restrict ourselves to the situations where these curves are coincident. In analogy with the behavior of other branes, if we take the k D6-branes to sit on top of each other, we expect that they can form a marginally bound state. In the classical description, we should then demand that the radius matrix (15) be proportional to the unit matrix. Given a choice of magnetic fluxes, this determines the F1 charge matrix Q_1 up to an overall multiplicative constant that parameterizes the radius of the combined system.

Since our matrices are all diagonal, the Born-Infeld action is unchanged except for the inclusion of an overall trace. Similarly, the energy is still given by $H = \int \mathrm{Tr}\,[Q_1 + Q_4 + Q_0]$.

[7] Quantum mechanically, we should demand that Tr Q_1 be an integer to ensure that the total number of F1 strings is integral.

Consider the example in which all k D6-branes are identical modulo the sign of $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$ so that both $\mathscr{F}_{1\theta}$ and $\mathscr{F}_{67}\mathscr{F}_{89}$ are proportional to the unit matrix.[8] Then, in terms of the total charges, the radius formula is

$$r^2 = \frac{1}{k^2\,(2\pi T_6)^2}\,\frac{Q_1^{\rm tot}\,(Q_4^{\rm tot})^2}{Q_0^{\rm tot}+Q_4^{\rm tot}}. \tag{18}$$

Observe that after fixing the conserved charges and imposing equal radii for the component tubes, there is still freedom in the values of the fluxes. These can be partially parameterized in terms of various non-conserved "charges," such as brane dipole moments. Due to the tubular configuration, our solution carries non-zero D6, D4, and D2 dipole moments, proportional to

$$\begin{aligned} Q_6^D &= T_6\; r\; k \\ Q_4^D &= T_6\; r\; \mathrm{Tr}\; \mathscr{F}_{67} \\ Q_2^D &= T_6\; r\; \mathrm{Tr}\; \mathscr{F}_{67}\; \mathscr{F}_{89} \equiv T_6\; r\; k_2. \end{aligned} \tag{19}$$

When the k D6-branes that form the tube are coincident, k_2 measures the local D2 brane dipole charge of the tube. It is also possible to see that both for a single tube, and for k tubes identical up to the sign of $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$, the dipole moments are related via:

$$\frac{Q_2^D}{Q_6^D} = \frac{k_2}{k} = \frac{Q_0^{\rm tot}}{Q_4^{\rm tot}}. \tag{20}$$

We will henceforth drop the superscripts on the $Q_p^{\rm tot}$ and denote them by Q_p. One can also derive the microscopic relation, (20), from the supergravity solutions that we construct in Sect. 3.4. In the supergravity solution, one has to set one of the three dipole charges to zero to obtain the solution with three asymptotic charges and two dipole charges. One then finds that (20) emerges from are careful examination of the near-horizon limit and the requirement that the solution be free of closed timelike curves [71].

If $\mathscr{F}_{67}$ and $\mathscr{F}_{89}$ are traceless, this tube has no D2 charge and no D4 dipole moment. More general tubes will not satisfy (20) and need not have vanishing D4 dipole moment when the D2 charge vanishes. We should also remark that the D2 dipole moment is an essential ingredient in constructing a supersymmetric three-charge tube of finite size. When this dipole moment goes to zero, the radius of the tube also becomes zero.

In general, we can construct a tube of arbitrary shape, and this tube will generically carry angular momentum in the (x^2,x^3) and (x^4,x^5) planes. We can also consider a round tube, made of k identical D6 branes wrapping an S^1 that lies for example in the (x^2,x^3) plane. The microscopic angular momentum density of such a configuration is given by the $(0,\theta)$ component of the energy-momentum tensor:

[8] One could also take $\mathrm{Tr}\mathscr{F}_{67} = \mathrm{Tr}\mathscr{F}_{89} = 0$ to cancel the D2 charge, but this does not affect the radius formula.

$$J_{23} = 2\pi \, r \, T_{0\theta} = 2\pi T_6 \, k \, r^2 \, \sqrt{\left(1+\mathscr{F}_{67}^2\right)\left(1+\mathscr{F}_{89}^2\right)}. \tag{21}$$

Now recall that supersymmetry requires $\mathscr{F}_{67} = \mathscr{F}_{89}$ and that $\mathrm{Tr}(\mathscr{F}_{67}\mathscr{F}_{89}) = Q_0/Q_4$, and so this may be rewritten as:

$$J_{23} = 2\pi T_6 \, k \, r^2 \, \left(1+\frac{Q_0}{Q_4}\right) = \frac{1}{2\pi T_6} \, \frac{Q_1 \, Q_4}{k}, \tag{22}$$

where we have used (18). It is interesting to note that this microscopic angular momentum density is not necessarily equal to the angular momentum measured at infinity. As we will see in the next section, from the full supergravity solution, the angular momenta of the three-charge supertube also have a piece coming from the supergravity fluxes. This is similar to the non-zero angular momentum coming from the Poynting vector, $\vec{E} \times \vec{B}$, in the static electromagnetic configuration consisting of an electron and a magnetic monopole [72].

Note also that when one adds D0 brane charge to a F1-D4 supertube, the angular momentum does not change, even if the radius becomes smaller. Hence, given charges of the same order, the angular momentum that the ring carries is of order the square of the charge (for a fixed number, k, of $D6$ branes). For more general three-charge supertubes, whose shape is an arbitrary curve inside $\mathbb{R}^4$, the angular momenta can be obtained rather straightforwardly from this shape by integrating the appropriate components of the BI energy-momentum tensor over the profile of the tube.

A T-duality along x^1 transforms our D0-D4-F1 tubes into the more familiar D1-D5-P configurations. This T-duality is implemented by the replacement $2\pi A^1 \to X^1$. The non-zero value of $\mathscr{F}_{1\theta}$ is translated by the T-duality into a non-zero value of $\partial_\theta X^1$. This means that the resulting D5-brane is in the shape of a helix whose axis is parallel to x^1. This is the same as the observation that the D2-brane supertube T-dualizes into a helical D1-brane. Since this helical shape is slightly less convenient to work with than a tube, we have chosen to emphasize the F1-D4-D0 description instead. Nevertheless, in the formulas that give the radius and angular momenta of the three-charge supertubes, we will use interchangingly the D1-D5-P and the D0-D4-F1 quantities, related via U-duality $N_0 \to N_p$, $N_4 \to N_5$, and $N_1 \to N_1$, with similar replacements for the Qs.

Exercise 3. *Write the combination of S-duality and T-duality transformations that correspond to this identification of the D1-D5-P and F1-D4-D0 quantities.*

2.3 Supertubes and Black Holes

The spinning three-charge black hole (also known as the BMPV black hole [73]) can only carry equal angular momenta, bounded above by:[9]

[9] In Sect. 3.4, we will re-derive the BMPV solution as part of a more complex solution. This bound can be seen from (1) and follows from the requirement that there are no closed time-like curves outside the horizon.

$$J_1^2 = J_2^2 \le N_1 N_5 N_P. \tag{23}$$

For the three-charge supertubes, the angular momenta are not restricted to be equal. A supertube configuration can have arbitrary shape, and carry any combination of the two angular momenta. For example, we can choose a closed curve such that the supertube cross-section lies in the (x^2, x^3) plane, for which $J_{23} \neq 0$ and $J_{45} = 0$. The bound on the angular momentum can be obtained from (22):

$$|J| = \frac{1}{2\pi T_6} \, \frac{Q_1 \, Q_4}{k} \le \frac{1}{2\pi T_6} \, Q_1 \, Q_4 = N_1 \, N_4, \tag{24}$$

where we have used $k \geq 1$ since it is the number of D6 branes. The quantized charges[10] are given by $Q_1 = \frac{1}{2\pi} N_1$, $Q_4 = (2\pi)^2 T_6 N_4$. We therefore see that a single D6 brane saturates the bound and that by varying the number of D6 branes or by appropriately changing the shape and orientation of the tube cross section, we can span the entire range of angular momenta between $-N_1 N_4$ and $+N_1 N_4$. Since (24) is quadratic the charges, one can easily exceed the black hole angular momentum bound in (23) by simply making Q_1 and Q_4 sufficiently large.

One can also compare the size of the supertube with the size of the black hole. Using (24), one can rewrite (18) in terms of the angular momentum:

$$r^2 = \frac{J^2}{Q_1 \, (Q_0 + Q_4)}, \tag{25}$$

Now recall that the tension of a D-brane varies as g_s^{-1} and that the charges, Q_0 and Q_4, appear in the Hamiltonian (14). This means that the quantization conditions on the D-brane charges must have the form $Q_j \sim N_j / g_s$. The energy of the fundamental string is independent of g_s and so $Q_1 \sim N_1$, with no factors of g_s. If we take $N_0 \approx N_1 \approx N_4 \approx N$ then we find:

$$r_{\text{tube}}^2 \sim g_s \, \frac{J^2}{N^2}. \tag{26}$$

From the BMPV black hole metric [73, 74], one can compute the proper length of the circumference of the horizon (as measured at one of the equator circles) to be

$$r_{\text{hole}}^2 \sim g_s \, \frac{N^3 - J^2}{N^2}. \tag{27}$$

The most important aspect of the (26) and (27) is that for comparable charges and angular momenta, the black hole and the three-charge supertube have comparable sizes. Moreover, these sizes grow with g_s in the same way. This is a very counter-intuitive behavior. Most of the objects we can think about tend to become smaller when gravity is made stronger and this is consistent with our intuition and the fact that gravity is an attractive force. The only "familiar" object that becomes larger with stronger gravity is a black hole. Nevertheless, three-charge supertubes also

[10] These charges are related to the charges that appear in the Hamiltonian by the corresponding tensions; more details about this can be found in [38].

become larger as gravity becomes stronger! The size of a tube is determined by a balance between the angular momentum of the system and the tension of the tubular brane. As the string coupling is increased, the D-brane tension decreases, and thus the size of the tube grows at exactly the same rate as the Schwarzschild radius of the black hole.[11]

This is the distinguishing feature that makes the three-charge supertubes (as well as the smooth geometries that we will obtain from their geometric transitions) unlike any other configuration that one counts in studying black hole entropy.

To be more precise, let us consider the counting of states that leads to the black hole entropy "à la Strominger and Vafa." One counts microscopic brane/string configurations at weak coupling where the system is of string scale in extent, and its Schwarzschild radius even smaller. One then imagines increasing the gravitational coupling; the Schwarzschild radius grows, becoming comparable to the size of the brane configuration at the "correspondence point" [75] and larger thereafter. When the Schwarzschild radius is much larger than the Planck scale, the system can be described as a black hole. There are thus two very different descriptions of the system: as a microscopic string theory object for small g_s, and as a black hole for large g_s. One then compares the entropy in the two regimes and finds an agreement, which is precise if supersymmetry forbids corrections during the extrapolation.

Three-charge supertubes behave differently. Their size grows at the same rate as the Schwarzschild radius, and thus they have no "correspondence point." Their description is valid in the same regime as the description of the black hole. If by counting such configurations one could reproduce the entropy of the black hole, then one should think about the supertubes as the large g_s continuation of the microstates counted at small g_s in the string/brane picture and therefore as the microstates of the corresponding black hole.

It is interesting to note that if the supertubes did not grow with *exactly* the same power of g_s as the black hole horizon, they would not be good candidates for being black hole microstates, and Mathur's conjecture would have been in some trouble. The fact that there exists a huge number of configurations that *do* have the same growth with g_s as the black hole is a non-trivial confirmation that these configurations may well represent black-hole microstates for the three-charge system.

We therefore expect that configurations constructed from three-charge supertubes will give us a large number of three-charge BPS black hole microstates. Nevertheless, we have seen that three-charge supertubes can have angular momenta larger than the BPS black hole and generically have $J_1 \neq J_2$. Hence, one can also ask if there exists a black object whose microstates those supertubes represent. In [38], it was conjectured that such an object should be a three-charge BPS black ring, despite the belief at the time that there was theorem that such BPS black rings could not exist. After more evidence for this conjecture came from the construction of

[11] Note that this is a feature only of three-charge supertubes; ordinary (two-charge) supertubes have a growth that is duality-frame dependent.

the flat limit of black rings [71], a gap in the proof of the theorem was found [76]. Subsequently the BPS black ring with equal charges and dipole charges was found in [77], followed by the rings with three arbitrary charges and three arbitrary dipole charges [68, 78, 79]. One of the morals of this story is that whenever one encounters an "established" result that contradicts intuition, one should really get to the bottom of it and find out why the intuition is wrong or to expose the cracks in established wisdom.

3 Black Rings and Supertubes

As we have seen in the D-brane analysis of the previous section, three-charge supertubes of arbitrary shape preserve the same supersymmetries as the three-charge black hole. Moreover, as we will see, three-charge supertube solutions that have three dipole charges can also have a horizon at large effective coupling, and thus become black rings. Therefore, one expects the existence of BPS configurations with an arbitrary distribution of black holes, black rings, and supertubes of arbitrary shape. Finding the complete supergravity solution for such configurations appears quite daunting. We now show that this is nevertheless possible and that the entire problem can be reduced to solving a linear system of equations in four-dimensional, Euclidean electromagnetism.

3.1 Supersymmetric Configurations

We begin by considering brane configurations that preserve the same supersymmetries as the three-charge black hole. In M-theory, the latter can be constructed by compactifying on a six-torus, T^6, and wrapping three sets of M2 branes on three orthogonal two-tori (see the first three rows of Table 1). Amazingly enough, one can

Table 1 Layout of the branes that give the supertubes and black rings in an M-theory duality frame

Brane	0	1	2	3	4	5	6	7	8	9	10
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$
M2	$\updownarrow$	$\star$	$\star$	$\star$	$\star$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\sigma)$			$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\sigma)$			$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$y^\mu(\sigma)$			$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\leftrightarrow$

Vertical arrows $\updownarrow$, indicate the directions along which the branes are extended, and horizontal arrows, $\leftrightarrow$, indicate the smearing directions. The functions, $y^\mu(\sigma)$, indicate that the brane wraps a simple closed curve in $\mathbb{R}^4$ that defines the black-ring or supertube profile. A star, $\star$, indicates that a brane is smeared along the supertube profile, and pointlike on the other three directions.

add a further three sets of M5 branes while preserving the same supersymmetries: Each set of M5 branes can be thought of as magnetically dual to a set of M2 branes in that the M5 branes wrap the four-torus, T^4, orthogonal to the T^2 wrapped by the M2 branes. The remaining spatial direction of the M5 branes follows a simple, closed curve, $y^\mu(\sigma)$, in the spatial section of the five-dimensional space-time. Since we wish to make a single, three-charge ring we take this curve to be the same for all three sets of M5 branes. This configuration is summarized in Table 1. In [68], it was argued that this was the most general three-charge brane configuration[12] consistent with the supersymmetries of the three-charge black-hole.

The metric corresponding to this brane configuration can be written as

$$\begin{aligned} ds_{11}^2 = ds_5^2 \,+\, &\big(Z_2 Z_3 Z_1^{-2}\big)^{\frac{1}{3}} (dx_5^2 + dx_6^2) \\ &+ \big(Z_1 Z_3 Z_2^{-2}\big)^{\frac{1}{3}} (dx_7^2 + dx_8^2) + \big(Z_1 Z_2 Z_3^{-2}\big)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2), \end{aligned} \tag{28}$$

where the five-dimensional space-time metric has the form:

$$ds_5^2 \equiv -\,(Z_1 Z_2 Z_3)^{-\frac{2}{3}} (dt + k)^2 + (Z_1 Z_2 Z_3)^{\frac{1}{3}} \; h_{\mu\nu} dy^\mu dy^\nu, \tag{29}$$

for some one-form field, k, defined upon the spatial section of this metric. Since we want the metric to be asymptotic to flat $\mathbb{R}^{4,1} \times T^6$, we require

$$ds_4^2 \equiv h_{\mu\nu} dy^\mu dy^\nu, \tag{30}$$

to limit to the flat, Euclidean metric on $\mathbb{R}^4$ at spatial infinity, and we require the warp factors, Z_I, to limit to constants at infinity. To fix the normalization of the corresponding Kaluza-Klein $U(1)$ gauge fields, we will take $Z_I \to 1$ at infinity.

The supersymmetry, ϵ, consistent with the brane configurations in Table 1 must satisfy:

$$\Big(\mathbb{1} - \Gamma^{056}\Big)\, \epsilon = \big(\mathbb{1} - \Gamma^{078}\big)\, \epsilon = \big(\mathbb{1} - \Gamma^{09\;10}\big)\, \epsilon = 0. \tag{31}$$

Since the product of all the gamma-matrices is the identity matrix, this implies

$$\big(\mathbb{1} - \Gamma^{1234}\big)\, \epsilon = 0, \tag{32}$$

which means that one of the four-dimensional helicity components of the four dimensional supersymmetry must vanish identically. The holonomy of the metric, (30), acting on the spinors is determined by

$$[\, \nabla_\mu \,,\, \nabla_\nu \,]\, \epsilon = \frac{1}{4}\, R^{(4)}_{\mu\nu cd}\, \Gamma^{cd}\, \epsilon, \tag{33}$$

where $R^{(4)}_{\mu\nu cd}$ is the Riemann tensor of (30). Observe that (33) vanishes identically as a consequence of (32) if the Riemann tensor is self-dual:

[12] Obviously one can choose add multiple curves and black hole sources.

$$R^{(4)}_{abcd} = \frac{1}{2}\, \varepsilon_{cd}{}^{ef}\, R^{(4)}_{abef}. \tag{34}$$

Such four-metrics are called "half-flat." Equivalently, note that the holonomy of a general Euclidean four-metric is $SU(2) \times SU(2)$ and that (34) implies that the holonomy lies only in one of these $SU(2)$ factors and that the metric is flat in the other factor. The condition (32) means that all the components of the supersymmetry upon which the non-trivial holonomy would act actually vanish. The other helicity components feel no holonomy and so the supersymmetry can be defined globally. One should also note that $SU(2)$ holonomy in four-dimensions is equivalent to requiring that the metric be hyper-Kähler.

Thus we can preserve the supersymmetry if and only if we take the four-metric to be hyper-Kähler. However, there is a theorem that states that any metric that is (i) Riemannian (signature $+4$) and regular, (ii) hyper-Kähler, and (iii) asymptotic to the flat metric on $\mathbb{R}^4$ *must be globally* the flat metric on $\mathbb{R}^4$. The obvious conclusion, which we will follow in this section, is that we simply take (30) to be the flat metric on $\mathbb{R}^4$. However, there are very important exceptions. First, we require the four-metric to be asymptotic to flat $\mathbb{R}^4$ because we want to interpret the object in asymptotically flat, five-dimensional space-time. If we want something that can be interpreted in terms of asymptotically flat, *four*-dimensional space-time then we want the four-metric to be asymptotic to the flat metric on $\mathbb{R}^3 \times S^1$. This allows for a lot more possibilities and includes the multi-Taub-NUT metrics [80]. Using such Taub-NUT metrics provides a straightforward technique for reducing the five-dimensional solutions to four dimensions [46, 81, 82, 83, 84].

The other exception will be the subject of subsequent sections of this review: The requirement that the four-metric be globally Riemannian is too stringent. As we will see, the metric can be allowed to change the overall sign since this can be compensated by a sign change in the warp factors of (29). In this section, however, we will suppose that the four-metric is simply that of flat $\mathbb{R}^4$.

3.2 The BPS Equations

The Maxwell three-form potential is given by

$$C^{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8 + A^{(3)} \wedge dx_9 \wedge dx_{10}, \tag{35}$$

where the six coordinates, x_A, parameterize the compactification torus, T^6, and $A^{(I)}$, $I = 1,\ 2,\ 3$, are one-form Maxwell potentials in the five-dimensional space-time and depend only upon the coordinates, y^μ, that parameterize the spatial directions. It is convenient to introduce the Maxwell "dipole field strengths," $\Theta^{(I)}$, obtained by removing the contributions of the electrostatic potentials

$$\Theta^{(I)} \equiv dA^{(I)} + d\big(Z_I^{-1}\, (dt + k)\big), \tag{36}$$

The most general supersymmetric configuration is then obtained by solving the *BPS equations*:

$$\Theta^{(I)} = \star_4 \Theta^{(I)}, \tag{37}$$

$$\nabla^2 Z_I = \frac{1}{2} C_{IJK} \star_4 \left(\Theta^{(J)} \wedge \Theta^{(K)} \right), \tag{38}$$

$$dk + \star_4 dk = Z_I \Theta^{(I)}, \tag{39}$$

where $\star_4$ is the Hodge dual taken with respect to the four-dimensional metric $h_{\mu\nu}$, and structure constants[13] are given by $C_{IJK} \equiv |\epsilon_{IJK}|$. It is important to note that if these equations are solved in the order presented above, then one is solving a linear system.

At each step in the solution-generating process one has the freedom to add homogeneous solutions of the equations. Since we are requiring that the fields fall off at infinity, this means that these homogeneous solutions must have sources in the base space, and since there is no topology in the $\mathbb{R}^4$ base, these sources must be singular. One begins by choosing the profiles, in $\mathbb{R}^4$, of the three types of M5 brane that source the $\Theta^{(I)}$. These fluxes then give rise to the explicit sources on the right-hand side of (38), but one also has the freedom to choose singular sources for (38) corresponding to the densities, $\rho_I(\sigma)$, of the three types of M2 branes. The M2 branes can be distributed at the same location as the M5 profile and can also be distributed away from this profile (see Fig. 2). The functions, Z_I, then appear in the final solution as warp factors and as the electrostatic potentials. There are thus two contributions to the total electric charge of the solution: The localized M2 brane sources described by $\rho_I(\sigma)$ and the induced charge from the fields, $\Theta^{(I)}$, generated by the M5 branes. It is in this sense that the solution contains electric charges that are dissolved in the fluxes generated by M5 branes, much like in the Klebanov-Strassler or Klebanov-Tseytlin solutions [85, 86].

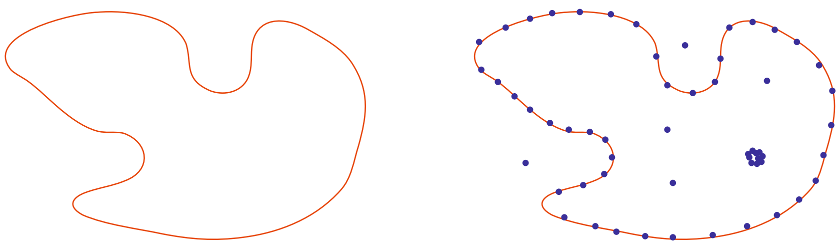

Fig. 2 The first two steps of the procedure to construct solutions. One first chooses an arbitrary M5 brane profile and then sprinkles the various types of M2 branes, either on the M5 brane profile or away from it. This gives a solution for an arbitrary superposition of black rings, supertubes, and black holes

[13] If the T^6 compactification manifold is replaced by a more general Calabi-Yau manifold, the C_{IJK} change accordingly.

The final step is to solve the last BPS equation, (39), which is sourced by a cross term between the magnetic and electric fields. Again there are homogeneous solutions that may need to be added and this time, however, they need to be adjusted so as to ensure that (29) has no closed time-like curves (CTC's). Roughly one must make sure that the angular momentum at each point does not exceed what can be supported by local energy density.

3.3 Asymptotic Charges

Even though a generic black ring is made from six sets of branes, there are only three conserved electric charges that can be measured from infinity. These are obtained from the three vector potentials, $A^{(I)}$, defined in (35), by integrating $\star_5 dA^{(I)}$ over the three-sphere at spatial infinity. Since the M5 branes run in a closed loop, they do not directly contribute to the electric charges. The electric charges are determined by electric fields at infinity and hence by the functions Z_I (36). Indeed, one has:

$$Z_I \sim 1 + c_1\, \frac{Q_I}{\rho^2} + \ldots\,, \qquad \rho \to \infty, \tag{40}$$

where c_1 is a normalization constant (discussed below), ρ is the standard, Euclidean radial coordinate in $\mathbb{R}^4$ and the Q_I are the electric charges. Note that while the M5 branes do not *directly* contribute to the electric charges, they do contribute indirectly via "charges dissolved in fluxes," that is, through the source terms on the right-hand side of (38).

To compute the angular momentum it is convenient to write the spatial $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$ and pass to two sets of polar coordinates, $(u,\ \theta_1)$ and $(v,\ \theta_2)$ in which the flat metric on $\mathbb{R}^4$ is:

$$ds_4^2 = \big(du^2 + u^2\, d\theta_1^2\big) + \big(dv^2 + v^2\, d\theta_2^2\big)\,. \tag{41}$$

There are two commuting angular momenta, J_1 and J_2, corresponding to the components of rotation in these two planes. One can then read off the angular momentum by making an expansion at infinity of the angular momentum vector, k, in (29):

$$k \sim c_2 \left(J_1\, \frac{u^2}{(u^2+v^2)^2} + J_2\, \frac{v^2}{(u^2+v^2)^2} \right) + \ldots\,, \qquad u, v \to \infty, \tag{42}$$

where c_2 is a normalization constant. The charges, Q_I, and the angular momenta, $J_1,\ J_2$, need to be correctly normalized in order to express them in terms of the quantized charges. The normalization depends upon the eleven-dimensional Planck length, ℓ_p, and the volume of the compactifying torus, T^6. The correct normalization can be found [68] and has been computed in many references. (For a good review, see [87].) Here we simply state that if L denotes the radius of the circles that make up the T^6 (so that the compactification volume is $V_6 = (2\pi L)^6$), then one obtains the canonically normalized quantities by using

$$c_1 = \frac{\ell_p^6}{L^4}, \qquad c_2 = \frac{\ell_p^9}{L^6}. \tag{43}$$

For simplicity, in most of the rest of this review we will take as system of units in which $\ell_p = 1$ and we will fix the torus volume so that $L = 1$. Thus one has $c_1 = c_2 = 1$.

3.4 An Example: A Three-Charge Black Ring with a Black Hole in the Middle

By solving the BPS equations, (37), (38) and (39), one can, in principle, find the supergravity solution for an arbitrary distribution of black rings and black holes. The metric for a general distribution of these objects will be extremely complicated, and so to illustrate the technique we will concentrate on a simpler system: A BMPV black hole at the center of a three-charge BPS black ring. An extensive review of black rings, both BPS and non-BPS can be found in [88]. Other interesting papers related to non-BPS black rings include [89, 90, 91, 92].

Since the ring sits in an $\mathbb{R}^2$ inside $\mathbb{R}^4$, it is natural to pass to the two sets of polar coordinates, $(u,\ \theta_1)$ and $(v,\ \theta_2)$ in which the base-space metric takes the form (41) We then locate the ring at $u = R$ and $v = 0$ and the black hole at $u = v = 0$.

The best coordinate system for actually solving the black ring equations is the one that has become relatively standard in the black-ring literature (see, for example, [77]). The change of variables is:

$$x = -\frac{u^2 + v^2 - R^2}{\sqrt{((u-R)^2 + v^2)((u+R)^2 + v^2)}}, \tag{44}$$

$$y = -\frac{u^2 + v^2 + R^2}{\sqrt{((u-R)^2 + v^2)((u+R)^2 + v^2)}}, \tag{45}$$

where $-1 \le x \le 1,\ -\infty < y \le -1$, and the ring is located at $y = -\infty$. This system has several advantages: it makes the electric and magnetic two-form field strengths sourced by the ring have a very simple form (see (47)), and it makes the ring look like a single point while maintaining separability of the Laplace equation. In these coordinates, the flat $\mathbb{R}^4$ metric has the form:

$$ds_4^2 = \frac{R^2}{(x-y)^2}\left(\frac{dy^2}{y^2-1} + \left(y^2 - 1\right) d\theta_1^2 + \frac{dx^2}{1-x^2} + \left(1 - x^2\right) d\theta_2^2\right). \tag{46}$$

The self-dual[14] field strengths that are sourced by the ring are then:

$$\Theta^{(I)} = 2\, q_i\, (dx \wedge d\theta_2 - dy \wedge d\theta_1). \tag{47}$$

[14] Our orientation is $\epsilon^{yx\theta_1\theta_2} = +1$.

The warp factors then have the form

$$Z_I = 1 + \frac{\overline{Q}_I}{R}(x-y) - \frac{2\, C_{IJK}\, q^J q^K}{R^2}\, \left(x^2 - y^2\right) - \frac{Y_I}{R^2}\, \frac{x-y}{x+y}, \tag{48}$$

and the angular momentum components are given by:

$$k_\psi = \left(y^2 - 1\right)\, g(x,y) - A\, (y+1), \qquad k_\phi = \left(x^2 - 1\right)\, g(x,y)\, ; \tag{49}$$

$$g(x,y) \equiv \left(\frac{C}{3}\, (x+y) + \frac{B}{2} - \frac{D}{R^2(x+y)} + \frac{K}{R^2(x+y)^2}\right) \tag{50}$$

where K represents the angular momentum of the BMPV black hole and

$$A \equiv 2\left(\sum q^I\right), \qquad B \equiv \frac{2}{R}\, \left(Q_I q^I\right), \tag{51}$$

$$C \equiv -\frac{8\, C_{IJK}\, q^I q^J q^K}{R^2}, \qquad D \equiv 2\, Y_I q^I. \tag{52}$$

The homogeneous solutions of (39) have already been chosen so as to remove any closed timelike curves (CTC).

The relation between the quantized ring and black-hole charges and the parameters appearing in the solution are:

$$\overline{Q}_I = \frac{\overline{N}_I\, \ell_p^6}{2L^4 R}, \qquad q^I = \frac{n^I\, \ell_p^3}{4L^2}, \qquad Y_I = \frac{N_I^{\rm BH}\, \ell_p^6}{L^4}, \qquad K = \frac{J^{\rm BMPV}\, \ell_p^9}{L^6}, \tag{53}$$

where L is the radius of the circles that make up the T^6 (so that $V_6 = (2\pi L)^6$) and ℓ_p is the eleven-dimensional Planck length.

As we indicated earlier, the asymptotic charges, N_I, of the solution are the sum of the microscopic charges on the black ring, $\overline{N}_I$, the charges of the black hole, $N_I^{\rm BH}$, and the charges dissolved in fluxes:

$$N_I = \overline{N}_I + N_I^{\rm BH} + \frac{1}{2}\, C_{IJK}\, n^J n^K. \tag{54}$$

Exercise 4. *Derive this expression for the charge from the asymptotic expansion of the Z_I in (48). Derive the relation between the parameters q^I and the quantized M5 charges n^I in (53) by integrating the magnetic M-theory four-form field strength around the ring profile. (See, for example, [87] in order to get the charge normalizations precisely correct.)*

The angular momenta of this solution are:

$$J_1 = J_\Delta + \left(\frac{1}{6}\, C_{IJK}\, n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + N_I^{\rm BH} n^I + J^{\rm BMPV}\right), \tag{55}$$

$$J_2 = -\left(\frac{1}{6}\, C_{IJK}\, n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + N_I^{\mathrm{BH}} n^I + J^{\mathrm{BMPV}}\right), \tag{56}$$

where

$$J_\Delta \equiv \frac{R^2 L^4}{l_p^6}\, \left(\sum n^I\right). \tag{57}$$

The entropy of the ring is:

$$S = \frac{2\pi A}{\kappa_{11}^2} = \pi\, \sqrt{\mathscr{M}} \tag{58}$$

where

$$\mathscr{M} \equiv 2\, n^1 n^2 \overline{N}_1 \overline{N}_2 + 2\, n^1 n^3 \overline{N}_1 \overline{N}_3 + 2\, n^2 n^3 \overline{N}_2 \overline{N}_3 - (n^1 \overline{N}_1)^2$$
$$- (n^2 \overline{N}_2)^2 - (n^3 \overline{N}_3)^2 - 4\, n^1 n^2 n^3\, J_T. \tag{59}$$

and

$$J_T \equiv J_\Delta + n^I N_I^{\mathrm{BH}} = \frac{R^2 L^4}{l_p^6}\, \left(\sum n^I\right) + n^I N_I^{\mathrm{BH}}. \tag{60}$$

As we will explain in more detail in Sect. 5.6, black rings can be related to four-dimensional black holes, and (59) is the square root of the $E_{7(7)}$ quartic invariant of the microscopic charges of the ring [56]; these microscopic charges are the n^I, the $\overline{N}_I$, and the angular momentum J_T. More generally, in configurations with multiple black rings and black holes, the quantity multiplying $n^1 n^2 n^3$ in $\mathscr{M}$ should be identified with the microscopic angular momentum of the ring. There are several ways to confirm that this identification is correct. First, one should note that J_T is the quantity that appears in the near-horizon limit of the metric and, in particular, determines the horizon area and hence entropy of the ring as in (58). This means that J_T is an intrinsic property of the ring. In the next section, we will discuss the process of lowering a black hole into the center of a ring and we will see, once again, that it is J_T that represents the intrinsic angular momentum of the ring.

The angular momenta of the solution may be re-written in terms of fundamental charges as:

$$J_1 = J_T + \left(\frac{1}{6}\, C_{IJK}\, n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + J^{\mathrm{BMPV}}\right)$$
$$J_2 = -\left(\frac{1}{6}\, C_{IJK}\, n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + N_I^{\mathrm{BH}} n^I + J^{\mathrm{BMPV}}\right). \tag{61}$$

Notice that in this form, J_1 contains no contribution coming from the combined effect of the electric field of the black hole and the magnetic field of the black ring. Such a contribution only appears in J_2.

3.5 Merging Black Holes and Black Rings

One can also use the methods above to study processes in which black holes and black rings are brought together and ultimately merge. Such processes are interesting in their own right, but we will also see later that they can be very useful in the study of microstate geometries.

It is fairly straightforward to generalize the solution of Sect. 3.4 to one that describes a black ring with a black hole on the axis of the ring but offset above the ring by a distance, $a = \alpha R$, where R is radius of the ring. (Both a and R are measured in the $\mathbb{R}^4$ base.) This is depicted in Fig. 3. The details of the exact solution may be found in [93], and we will only summarize the main results here.

The total charge of the combined system is independent of α and is given by (54). Similarly, the entropy of the black ring is still given by (58) and (59) but now with J_T defined by:

$$J_T = J_\Delta + \frac{n^I N_I^{\rm BH}}{1+\alpha^2} \equiv \frac{R^2 L^4}{l_p^6}\ \left(\sum n^I\right) + \frac{n^I N_I^{\rm BH}}{1+\alpha^2}. \tag{62}$$

The horizon area of the black hole is unmodified by the presence of the black ring and, in particular, its dependence on α only comes via J_T. Thus, for an adiabatic process, the quantity, $\mathscr{M}$, in (59) must remain fixed, and therefore J_T must remain fixed. This is consistent with identifying J_T as the intrinsic angular momentum of the ring.

The two angular momenta of the system are:

$$J_1 = J_T + \left(\frac{1}{6}\, C_{IJK} n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + J^{\rm BMPV}\right), \tag{63}$$

$$J_2 = -\left(\frac{1}{6}\, C_{IJK} n^I n^J n^K + \frac{1}{2}\, \overline{N}_I n^I + \frac{N_I^{\rm BH} n^I}{1+\alpha^2} + J^{\rm BMPV}\right). \tag{64}$$

If we change the separation of the black hole and black ring while preserving the axial symmetry, that is, if we vary α, then the symmetry requires J_1 to be conserved. Once again we see that this means that J_T must remain fixed.

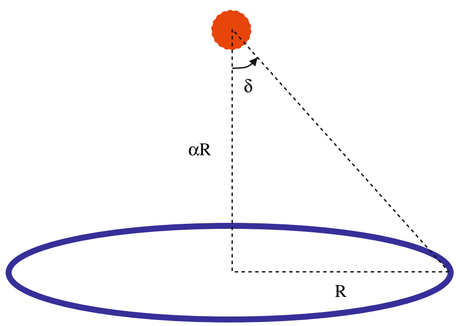

Fig. 3 The configuration black ring with an off-set black hole on its axis. The parameter, α, is related to the angle of approach, δ, by $\alpha \equiv \cot\delta$

The constancy of J_T along with (62) imply that as the black hole is brought near the black ring, the embedding radius of the latter, R, must change according to:

$$R^2 = \frac{l_p^6}{L^4} \left(\sum n^I\right)^{-1} \left(J_T - \frac{n^I N_I^{\rm BH}}{1+\alpha^2}\right). \tag{65}$$

For fixed microscopic charges this formula gives the radius of the ring as a function of the parameter α. The black hole will merge with the black ring if and only if R vanishes for some value of α. That is, if and only if

$$J_T \le n^I N_I^{\rm BH}. \tag{66}$$

The vanishing of R suggests that the ring is pinching off; however, in the physical metric, (29), the ring generically has finite size as it settles onto the horizon of the black hole. Indeed, the value of $\alpha = \tan\zeta$ at the merger determines the latitude, ζ, at which the ring settles on the black hole. If it occurs at $\alpha = 0$ then the ring merges by grazing the black hole at the equator.

At merger $(R=0)$, one can see that $J_1 = J_2$ and so the resulting object will have $J_1 = J_2$ given by (63). This will be a BMPV black hole, and its electric charges are simply given by (54). We can therefore use (1) to determine the final entropy after the merger. Note that the process we are considering is adiabatic up to the point where the ring touches the horizon of the black hole. The process of swallowing the ring is not necessarily adiabatic, but we assume that the black hole does indeed swallow the black ring, and we can then compute the entropy from the charges and angular momentum of the resulting BMPV black hole.

In general, the merger of a black hole and a black ring is irreversible, that is, the total horizon area increases in the process. However, there is precisely one situation in which the merger is reversible, and that requires *all* of the following to be true:

1. The ring must have zero horizon area (with a slight abuse of terminology we will also refer to such rings as *supertubes*).
2. The black hole that one begins with must have zero horizon area, i.e. it must be *maximally spinning*.
3. The ring must meet the black hole by grazing it at the equator.
4. There are two integers, $\bar{P}$ and P^{BH} such that

$$\overline{N}_I = \frac{\overline{P}}{n^I} \qquad \text{and} \qquad N_I^{BH} = \frac{P^{BH}}{n^I}, \qquad I = 1,2,3. \tag{67}$$

If all of these conditions are met then the end result is also a maximally spinning BMPV black hole and hence also has zero horizon area.

Note that the last condition implies that

$$N_I \equiv \overline{N}_I + \frac{1}{2}\, C_{IJK}\, n^J\, n^K = \frac{(\overline{P} + n^1\, n^2\, n^3)}{n^I}, \tag{68}$$

and therefore the electric charges of black ring *and* its charges dissolved in fluxes $(\frac{1}{2} C_{IJK}\, n^J\, n^K)$ must both be aligned exactly parallel to the electric charges of the black hole. Conversely, if conditions 1–3 are satisfied, but the charge vectors of the black hole and black ring are *not parallel* then the merger will be irreversible. This observation will be important in Sect. 8.

4 Geometric Interlude: Four-Dimensional Black Holes and Five-Dimensional Foam

In Sect. 3.1, we observed that supersymmetry allows us to take the base-space metric to be any hyper-Kähler metric. There are certainly quite a number of interesting four-dimensional hyper-Kähler metrics, and in particular, there are the multi-centered Gibbons-Hawking metrics. These provide examples of asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) spaces, which are asymptotic to $\mathbb{R}^4/\mathbb{Z}_n$ and $\mathbb{R}^3 \times S^1$, respectively. Using ALF metrics provides a smooth way to transition between a five-dimensional and a four-dimensional interpretation of a certain configuration. Indeed, the size of the S^1 is usually a modulus of a solution, and thus is freely adjustable. When this size is large compared to the size of the source configuration, this configuration is essentially five-dimensional; if the S^1 is small, then the configuration has a four-dimensional description.

We noted earlier that a regular, Riemannian, hyper-Kähler metric that is asymptotic to flat $\mathbb{R}^4$ is necessarily flat $\mathbb{R}^4$ *globally*. The non-trivial ALE metrics get around this by having a discrete identification at infinity but, as a result, do not have an asymptotic structure that lends itself to a space-time interpretation. However, there is an unwarranted assumption here: One should remember that the goal is for the five-metric (29) to be regular and Lorentzian, and this might be achievable if singularities of the four-dimensional base space were canceled by the warp factors. More specifically, we are going to consider base-space metrics (30) whose overall sign is allowed to change in interior regions. That is, we are going to allow the signature to flip from $+4$ to -4. We will call such metrics *ambipolar*.

The potentially singular regions could actually be regular if the warp factors, Z_I, all flip sign whenever the four-metric signature flips. Indeed, we suspect that the desired property may follow quite generally from the BPS equations through the four-dimensional dualization on the right-hand side of (38). Obviously, there are quite a number of details to be checked before complete regularity is proven, but we will see below that this can be done for ambipolar Gibbons-Hawking metrics.

Because of these two important applications, we now give a review of Gibbons-Hawking geometries [80, 94] and their elementary ambipolar generalization. These metrics have the virtue of being simple enough for very explicit computation and yet capture some extremely interesting physics.

4.1 Gibbons-Hawking Metrics

Gibbons-Hawking metrics have the form of a $U(1)$ fibration over a flat $\mathbb{R}^3$ base:

$$h_{\mu\nu}dx^\mu dx^\nu = V^{-1}\,\left(d\psi + \vec{A}\cdot d\vec{y}\right)^2 + V\,\left(dx^2+dy^2+dz^2\right), \tag{69}$$

where we write $\vec{y} = (x,\ y,\ z)$. The function, V, is harmonic on the flat $\mathbb{R}^3$ while the connection, $A = \vec{A}\cdot d\vec{y}$, is related to V via

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}V. \tag{70}$$

This family of metrics is the unique set of hyper-Kähler metrics with a tri-holomorphic $U(1)$ isometry.[15] Moreover, four-dimensional hyper-Kähler manifolds with $U(1)\times U(1)$ symmetry must, at least locally, be Gibbons-Hawking metrics with an extra $U(1)$ symmetry around an axis in the $\mathbb{R}^3$ [95].

In the standard form of the Gibbons-Hawking metrics, one takes V to have a finite set of isolated sources. That is, let $\vec{y}^{(j)}$ be the positions of the source points in the $\mathbb{R}^3$ and let $r_j \equiv |\vec{y}-\vec{y}^{(j)}|$. Then one takes:

$$V = \varepsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}, \tag{71}$$

where one usually takes $q_j \geq 0$ to ensure that the metric is Riemannian (positive definite). We will later relax this restriction. There appear to be singularities in the metric at $r_j = 0$; however, if one changes to polar coordinates centered at $r_j = 0$ with radial coordinate to $\rho = 2\sqrt{|\vec{y}-\vec{y}^{(j)}|}$, then the metric is locally of the form:

$$ds_4^2 \sim d\rho^2 + \rho^2\; d\Omega_3^2, \tag{72}$$

where $d\Omega_3^2$ is the standard metric on $S^3/\mathbb{Z}_{|q_j|}$. In particular, this means that one must have $q_j \in \mathbb{Z}$ and if $|q_j| = 1$ then the space looks locally like $\mathbb{R}^4$. If $|q_j| \neq 1$, then there is an orbifold singularity, but since this is benign in string theory, we will view such backgrounds as regular.

If $\varepsilon_0 \neq 0$, then $V \to \varepsilon_0$ at infinity and so the metric (69) is asymptotic to flat $\mathbb{R}^3 \times S^1$, that is, the base is asymptotically locally flat (ALF). The five-dimensional space-time is thus asymptotically compactified to a four-dimensional space-time. This a standard Kaluza-Klein reduction and the gauge field, $\vec{A}$, yields a non-trivial, four-dimensional Maxwell field whose sources, from the ten-dimensional perspective, are simply D6 branes. In Sect. 5.6 we will make extensive use of of the fact that introducing a constant term into V yields a further compactification, and through this we can relate five-dimensional physics to four-dimensional physics.

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[15] Tri-holomorphic means that the $U(1)$ preserves all three complex structures of the hyper-Kähler metric.

Now suppose that one has $\varepsilon_0 = 0$. At infinity, in $\mathbb{R}^3$ one has $V \sim q_0/r$, where $r \equiv |\vec{y}|$ and

$$q_0 \equiv \sum_{j=1}^{N} q_j. \tag{73}$$

Hence spatial infinity in the Gibbons-Hawking metric also has the form (72), where

$$r = \frac{1}{4}\rho^2, \tag{74}$$

and $d\Omega_3^2$ is the standard metric on $S^3/\mathbb{Z}_{|q_0|}$. For the Gibbons-Hawking metric to be asymptotic to the positive definite, flat metric on $\mathbb{R}^4$ one must have $q_0 = 1$. Note that for the Gibbons-Hawking metrics to be globally positive definite, one would also have to take $q_j \geq 0$ and thus the only such metric would have to have $V \equiv \frac{1}{r}$. The metric (69) is then the flat metric on $\mathbb{R}^4$ globally, as can be seen by using the change of variables (74). The only way to get non-trivial metrics that are asymptotic to flat $\mathbb{R}^4$ is by taking some of the $q_j \in \mathbb{Z}$ to be negative.

4.2 Homology and Cohomology

The multi-center Gibbons-Hawking (GH) metrics also contain $\frac{1}{2}N(N-1)$ topologically non-trivial two-cycles, Δ_{ij}, that run between the GH centers. These two-cycles can be defined by taking any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ and considering the $U(1)$ fiber of (69) along the curve. This fiber collapses to zero at the GH centers, and so the curve and the fiber sweep out a 2-sphere (up to $\mathbb{Z}_{|q_j|}$ orbifolds). See Fig. 4. These spheres intersect one another at the common points $\vec{y}^{(j)}$. There are $(N-1)$ linearly independent homology two-spheres, and the set $\Delta_{i(i+1)}$ represents a basis.[16]

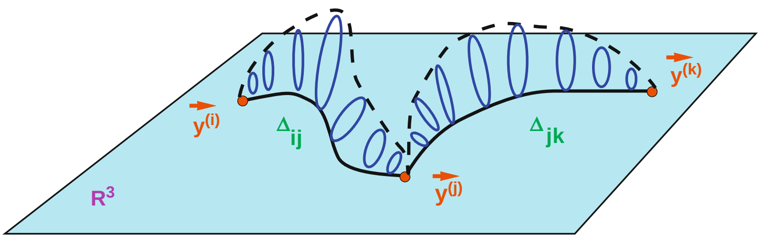

Fig. 4 This figure depicts some non-trivial cycles of the Gibbons-Hawking geometry. The behavior of the $U(1)$ fiber is shown along curves between the sources of the potential, V. Here the fibers sweep out a pair of intersecting homology spheres

[16] The integer homology corresponds to the root lattice of $SU(N)$ with an intersection matrix given by the inner product of the roots.

It is also convenient to introduce a set of frames

$$\hat{e}^1 = V^{-\frac{1}{2}}\,(d\psi + A), \qquad \hat{e}^{a+1} = V^{\frac{1}{2}}\,dy^a, \quad a = 1,2,3. \tag{75}$$

and two associated sets of two-forms:

$$\Omega_{\pm}^{(a)} \equiv \hat{e}^1 \wedge \hat{e}^{a+1} \pm \frac{1}{2}\,\epsilon_{abc}\,\hat{e}^{b+1} \wedge \hat{e}^{c+1}, \qquad a = 1,2,3. \tag{76}$$

The two-forms, $\Omega_{-}^{(a)}$, are anti-self-dual, harmonic and non-normalizable, and they define the hyper-Kähler structure on the base. The forms, $\Omega_{+}^{(a)}$, are self-dual and can be used to construct harmonic fluxes that are dual to the two-cycles. Consider the self-dual two-form:

$$\Theta \equiv \sum_{a=1}^{3} \big(\partial_a\big(V^{-1}\,H\big)\big)\;\Omega_{+}^{(a)}. \tag{77}$$

Then Θ is closed (and hence co-closed and harmonic) if and only if H is harmonic in $\mathbb{R}^3$, i.e. $\nabla^2 H = 0$. We now have the choice of how to distribute sources of H throughout the $\mathbb{R}^3$ base of the GH space; such a distribution may correspond to having multiple black rings and black holes in this space. Nevertheless, if we want to obtain a geometry that has no singularities and no horizons, Θ has to be regular, and this happens if and only if H/V is regular; this occurs if and only if H has the form:

$$H = h_0 + \sum_{j=1}^{N} \frac{h_j}{r_j}. \tag{78}$$

Also note that the "gauge transformation"

$$H \to H + c\,V, \tag{79}$$

for some constant, c, leaves Θ unchanged, and so there are only N independent parameters in H. In addition, if $\varepsilon = 0$ then one must take $h_0 = 0$ for Θ to remain finite at infinity. The remaining $(N-1)$ parameters then describe harmonic forms that are dual to the non-trivial two-cycles. If $\varepsilon \neq 0$ then the extra parameter is that of a Maxwell field whose gauge potential gives the Wilson line around the S^1 at infinity.

Exercise 5. *Show that the two-form, Θ, defined by (77) and (78) is normalizable on standard GH spaces (with $V > 0$ everywhere). That is, show that Θ square integrable:*

$$\int \Theta \wedge \Theta < \infty, \tag{80}$$

where the integral is taken of the whole GH base space.

It is straightforward to find a *local* potential such that $\Theta = dB$:

$$B \equiv V^{-1}\,H\,(d\psi + A) + \vec{\xi}\cdot d\vec{y}, \tag{81}$$

where

$$\vec{\nabla} \times \vec{\xi} = -\vec{\nabla} H. \tag{82}$$

Hence, $\vec{\xi}$ is a vector potential for magnetic monopoles located at the singular points of H.

To determine how these fluxes thread the two-cycles, we need the explicit forms for the vector potential, B, and to find these we first need the vector fields, $\vec{v}_i$, that satisfy:

$$\vec{\nabla} \times \vec{v}_i = \vec{\nabla}\left(\frac{1}{r_i}\right). \tag{83}$$

One then has:

$$\vec{A} = \sum_{j=1}^{N} q_j\,\vec{v}_j, \qquad \vec{\xi} = \sum_{j=1}^{N} h_j\,\vec{v}_j. \tag{84}$$

If we choose coordinates so that $\vec{y}^{(i)} = (0,0,a)$ and let ϕ denote the polar angle in the $(x,\ y)$-plane, then:

$$\vec{v}_i \cdot d\vec{y} = \left(\frac{(z-a)}{r_i} + c_i\right) d\phi, \tag{85}$$

where c_i is a constant. The vector field, $\vec{v}_i$, is regular away from the z-axis but has a Dirac string along the z-axis. By choosing c_i, we can cancel the string along the positive or negative z-axis, and by moving the axis we can arrange these strings to run in any direction we choose, but they must start or finish at some $\vec{y}^{(i)}$, or run out to infinity.

Now consider what happens to B in the neighborhood of $\vec{y}^{(i)}$. Since the circles swept out by ψ and ϕ are shrinking to zero size, the string singularities near $\vec{y}^{(i)}$ are of the form:

$$B \sim \frac{h_i}{q_i}\left(d\psi + q_i\left(\frac{(z-a)}{r_i} + c_i\right) d\phi\right) - h_i\left(\frac{(z-a)}{r_i} + c_i\right) d\phi \sim \frac{h_i}{q_i}\, d\psi. \tag{86}$$

This shows that the vector, $\vec{\xi}$, in (81) cancels the string singularities in the $\mathbb{R}^3$. The singular components of B thus point along the $U(1)$ fiber of the GH metric.

Choose any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ and define the two-cycle, Δ_{ij}, as in Fig. 4. If one has $V > 0$ then the vector field, B, is regular over the whole of Δ_{ij} except at the end-points, $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Let $\widehat{\Delta}_{ij}$ be the cycle Δ_{ij} with the poles excised. Since Θ is regular at the poles, then the expression for the flux, Π_{ij}, through Δ_{ij} can be obtained as follows:

$$\begin{aligned}\Pi_{ij} &\equiv \frac{1}{4\,\pi}\int_{\Delta_{ij}} \Theta = \frac{1}{4\,\pi}\int_{\widehat{\Delta}_{ij}} \Theta = \frac{1}{4\,\pi}\int_{\partial\widehat{\Delta}_{ij}} B \\ &= \frac{1}{4\,\pi}\int_0^{4\pi} d\psi\,\left(B|_{y^{(j)}} - B|_{y^{(i)}}\right) = \left(\frac{h_j}{q_j} - \frac{h_i}{q_i}\right).\end{aligned} \tag{87}$$

We have normalized these periods for later convenience.

On an ambipolar GH space where the cycle runs between positive and negative GH points, the flux, Θ, and the potential B are both singular when $V = 0$, and so this integral is a rather formal object. However, we will see in Sect. 6.3 that when we extend to the five-dimensional metric, the physical flux of the complete Maxwell field combines Θ with another term so that the result is completely regular. Moreover, the physical flux through the cycle is still given by (87). We will therefore refer to (87) as the magnetic flux even in ambipolar metrics, and we will see that such fluxes are directly responsible for holding up the cycles

5 Solutions on a Gibbons-Hawking Base

5.1 Solving the BPS Equations

Our task now is to solve the BPS equations (37), (38), (39) but now with a Gibbons-Hawking base metric. Such solutions have been derived before for positive-definite Gibbons-Hawking metrics [79, 96], and it is trivial to generalize to the ambipolar form. For the present, we will not impose any conditions on the sources of the BPS equations.

In Sect. 4.2, we saw that there was a simple way to obtain self-dual two-forms, $\Theta^{(I)}$, that satisfy (37). That is, we introduce three harmonic functions, K^I, on $\mathbb{R}^3$ that satisfy $\nabla^2 K^I = 0$, and define $\Theta^{(I)}$ as in (77) by replacing H with K^I. We will not, as yet, assume any specific form for K^I.

Exercise 6. *Substitute these two-forms into (38) and show that the resulting equation has the solution:*

$$Z_I = \frac{1}{2}\, C_{IJK}\, V^{-1}\, K^J K^K + L_I, \tag{88}$$

where the L_I are three more independent harmonic functions.

We now write the one-form, k, as:

$$k = \mu\, (d\psi + A) + \omega \tag{89}$$

and then (39) becomes:

$$\vec{\nabla} \times \vec{\omega} = (V\vec{\nabla}\mu - \mu\vec{\nabla}V) - V\, \sum_{I=1}^{3} Z_I\, \vec{\nabla}\left(\frac{K^I}{V}\right). \tag{90}$$

Taking the divergence yields the following equation for μ:

$$\nabla^2\mu = V^{-1}\, \vec{\nabla}\cdot\left(V \sum_{I=1}^{3} Z_I \vec{\nabla}\frac{K^I}{V}\right), \tag{91}$$

which is solved by:

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2V} K^I L_I + M, \tag{92}$$

where M is yet another harmonic function on $\mathbb{R}^3$. Indeed, M determines the anti-self-dual part of dk that cancels out of (39). Substituting this result for μ into (90) we find that ω satisfies:

$$\vec{\nabla} \times \vec{\omega} = V \vec{\nabla} M - M \vec{\nabla} V + \frac{1}{2} \left(K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I \right). \tag{93}$$

The integrability condition for this equation is simply the fact that the divergence of both sides vanish, which is true because K^I, L_I, M, and V are harmonic.

5.2 Some Properties of the Solution

The solution is thus characterized by the harmonic functions K^I, L_I, V, and M. The gauge invariance, (79), extends in a straightforward manner to the complete solution:

$$\begin{aligned} K^I &\to K^I + c^I V, \\ L_I &\to L_I - C_{IJK} c^J K^K - \frac{1}{2} C_{IJK} c^J c^K V, \\ M &\to M - \frac{1}{2} c^I L_I + \frac{1}{12} C_{IJK} \left(V c^I c^J c^K + 3 c^I c^J K^K \right), \end{aligned} \tag{94}$$

where the c^I are three arbitrary constants.[17]

The eight functions that give the solution may also be identified with the eight independent parameters in the **56** of the $E_{7(7)}$ duality group in four dimensions:

$$\begin{aligned} x_{12} &= L_1, & x_{34} &= L_2, & x_{56} &= L_3, & x_{78} &= -V, \\ y_{12} &= K^1, & y_{34} &= K^2, & y_{56} &= K^3, & y_{78} &= 2M. \end{aligned} \tag{95}$$

With these identifications, the right-hand side of (93) is the symplectic invariant of the **56** of $E_{7(7)}$:

$$\vec{\nabla} \times \vec{\omega} = \frac{1}{4} \sum_{A,B=1}^{8} (y_{AB} \vec{\nabla} x_{AB} - x_{AB} \vec{\nabla} y_{AB}). \tag{96}$$

---

[17] Note that this gauge invariance exists for any C_{IJK}, not only for those coming from reducing M-theory on T^6.

We also note that the quartic invariant of the **56** of $E_{7(7)}$ is determined by:

$$\begin{aligned} J_4 = & -\frac{1}{4}(x_{12}y^{12}+x_{34}y^{34}+x_{56}y^{56}+x_{78}y^{78})^2 - x_{12}x_{34}x_{56}x_{78} \\ & - y^{12}y^{34}y^{56}y^{78} + x_{12}x_{34}y^{12}y^{34} + x_{12}x_{56}y^{12}y^{56} + x_{34}x_{56}y^{34}y^{56} \\ & + x_{12}x_{78}y^{12}y^{78} + x_{34}x_{78}y^{34}y^{78} + x_{56}x_{78}y^{56}y^{78}, \end{aligned} \tag{97}$$

and we will see that this plays a direct role in the expression for the scale of the $U(1)$ fibration. It also plays a central role in the expression for the horizon area of a four-dimensional black hole [97].

In principle, we can choose the harmonic functions K^I, L_I, and M to have sources that are localized anywhere on the base. These solutions then have localized brane sources, and include, for example, supertubes and black rings in Taub-NUT [46, 82, 83, 84], which we will review in Sect. 5.5. Such solutions also include more general multi-center black hole configurations in four dimensions, of the type considered by Denef and collaborators [98, 99, 100].

Nevertheless, our focus for the moment is on obtaining smooth horizonless solutions, which correspond to microstates of black holes and black rings, and we choose the harmonic functions so that there are no brane charges anywhere, and all the charges come from the smooth cohomological fluxes that thread the non-trivial cycles.

5.3 Closed Time-Like Curves

To look for the presence of closed time-like curves in the metric, one considers the space-space components of the metric given by (28), (29), and (69). That is, one goes to the space-like slices obtained by taking t to be a constant. The T^6 directions immediately yield the requirement that $Z_I Z_J > 0$, while the metric on the four-dimensional base reduces to:

$$\begin{aligned} ds_4^2 = & -W^{-4}\,\big(\mu(d\psi+A)+\omega\big)^2 \\ & +W^2V^{-1}\big(d\psi+A\big)^2 + W^2V\big(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2\big), \end{aligned} \tag{98}$$

where we have chosen to write the metric on $\mathbb{R}^3$ in terms of a generic set of spherical polar coordinates, (r,θ,ϕ) and where we have defined the warp-factor, W, by:

$$W \equiv (Z_1\, Z_2\, Z_3)^{1/6}. \tag{99}$$

There is some potentially singular behavior arising from the fact that the Z_I, and hence W, diverge on the locus, $V=0$ (see (88)). However, one can show that if one expands the metric (98) and uses the expression, (92), then all the dangerous divergent terms cancel and the metric is regular. We will discuss this further below and in Sect. 5.4.

Expanding (98) leads to:

$$
\begin{aligned}
ds_4^2 &= W^{-4}\,(W^6V^{-1}-\mu^2)\left(d\psi + A - \frac{\mu\,\omega}{W^6V^{-1}-\mu^2}\right)^2 - \frac{W^2\,V^{-1}}{W^6V^{-1}-\mu^2}\,\omega^2 \\
&\quad + W^2V\big(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\big) \\
&= \frac{\mathcal{Q}}{W^4V^2}\left(d\psi + A - \frac{\mu\,V^2}{\mathcal{Q}}\,\omega\right)^2 + W^2V\left(r^2\sin^2\theta\, d\phi^2 - \frac{\omega^2}{\mathcal{Q}}\right) \\
&\quad + W^2V(dr^2 + r^2 d\theta^2),
\end{aligned} \tag{100}
$$

where we have introduced the quantity:

$$
\mathcal{Q} \equiv W^6\,V - \mu^2\,V^2 = Z_1Z_2Z_3V - \mu^2\,V^2. \tag{101}
$$

Upon evaluating $\mathcal{Q}$ as a function of the harmonic functions that determine the solution, one obtains a beautiful result:

$$
\begin{aligned}
\mathcal{Q} = &-M^2\,V^2 - \frac{1}{3}\,M\,C_{IJK}K^I\,K^J\,K^k - M\,V\,K^I\,L_I - \frac{1}{4}\,(K^IL_I)^2 \\
&+ \frac{1}{6}\,VC^{IJK}L_IL_JL_K + \frac{1}{4}\,C^{IJK}C_{IMN}L_JL_KK^MK^N
\end{aligned} \tag{102}
$$

with $C^{IJK} \equiv C_{IJK}$. We can straightforwardly see that when we consider M-theory compactified on T^6, then $C^{IJK} = |\epsilon^{IJK}|$, and $\mathcal{Q}$ is nothing other than the $E_{7(7)}$ quartic invariant (97) where the xs and ys are identified as in (95). This is expected from the fact that the solutions on a GH base have an extra $U(1)$ invariance and hence can be thought of as four-dimensional. The four-dimensional supergravity obtained by compactifying M-theory on T^7 is $N=8$ supergravity, which has an $E_{7(7)}$ symmetry group. Of course, the analysis above and in particular equation (102) are valid for solutions of arbitrary five-dimensional $U(1)^N$ ungauged supergravities on a GH base. More details on the explicit relation for general theories can be found in [101].

Exercise 7. *Check that $\mathcal{Q}$ is invariant under the gauge transformation (94)*

Observe that (100) only involves V in the combinations W^2V and $\mathcal{Q}$ and both of these are regular as $V \to 0$. Thus, at least the spatial metric is regular at $V=0$. In Sect. 5.4, we will show that the complete solution is regular as one passes across the surface $V=0$.

From (100) and (28), we see that to avoid CTCs, the following inequalities must be true everywhere:

$$
\mathcal{Q} \ge 0, \quad W^2\,V \ge 0, \quad \big(Z_J\,Z_K\,Z_I^{-2}\big)^{\frac{1}{3}} = W^2 Z_I^{-1} \ge 0, \;\; I = 1,2,3. \tag{103}
$$

The last two conditions can be subsumed into:

$$
V\,Z_I = \frac{1}{2}\,C_{IJK}\,K^J\,K^K + L_I\,V \ge 0, \qquad I = 1,2,3. \tag{104}
$$

The obvious danger arises when V is negative. We will show in the next sub-section that all these quantities remain finite and positive in a neighborhood of $V = 0$, despite the fact that W blows up. Nevertheless, these quantities could possibly be negative away from the $V = 0$ surface. While we will, by no means, make a complete analysis of the positivity of these quantities, we will discuss it further in Sect. 6.5, and show that (104) does not present a significant problem in a simple example. One should also note that $\mathcal{Q} \geq 0$ requires $\prod_I (VZ_I) \geq \mu^2 V^4$, and so, given (104), the constraint $\mathcal{Q} \geq 0$ is still somewhat stronger.

Also note that there is a danger of CTCs arising from Dirac-Misner strings in ω. That is, near $\theta = 0,\ \pi$ the $-\omega^2$ term could be dominant unless ω vanishes on the polar axis. We will analyze this issue completely, when we consider bubbled geometries in Sect. 6.

Finally, one can also try to argue [48] that the complete metric is stably causal and that the t coordinate provides a global time function [102]. In particular, t will then be monotonic increasing on future-directed non-space-like curves, and hence there can be no CTCs. The coordinate t is a time function if and only if

$$-g^{\mu\nu}\partial_\mu t\, \partial_\nu t = -g^{tt} = (W^2 V)^{-1}(\mathcal{Q} - \omega^2) > 0, \tag{105}$$

where ω is squared using the $\mathbb{R}^3$ metric. This is obviously a slightly stronger condition than $\mathcal{Q} \geq 0$ in (103).

5.4 Regularity of the Solution and Critical Surfaces

As we have seen, the general solutions we will consider have functions, V, that change sign on the $\mathbb{R}^3$ base of the GH metric. Our purpose here is to show that such solutions are completely regular, with positive definite metrics, in the regions where V changes sign. As we will see, the "critical surfaces" where V vanishes are simply a set of completely harmless, regular hypersurfaces in the full five-dimensional geometry.

The most obvious issue is that if V changes sign, then the overall sign of the metric (69) changes, and there might be whole regions of closed time-like curves when $V < 0$. However, we remarked above that the warp factors, in the form of W, prevent this from happening. Specifically, the expanded form of the complete, eleven-dimensional metric when projected onto the GH base yields (100). Moreover

$$W^2\, V = \left(Z_1\, Z_2\, Z_3\, V^3\right)^{\frac{1}{3}} \sim \left((K_1\, K_2\, K_3)^2\right)^{\frac{1}{3}} \tag{106}$$

on the surface $V = 0$. Hence $W^2 V$ is regular and positive on this surface, and therefore the space-space part (100) of the full eleven-dimensional metric is regular.

There is still the danger of singularities at $V = 0$ for the other background fields. We first note that there is no danger of such singularities being hidden implicitly in the $\vec{\omega}$ terms. Even though (90) suggests that the source of $\vec{\omega}$ is singular at $V = 0$, we see from (93) that the source is regular at $V = 0$, and thus there is nothing hidden in $\vec{\omega}$. We therefore need to focus on the explicit inverse powers of V in the solution.

The factors of V cancel in the torus warp factors, which are of the form $(Z_I Z_J Z_K^{-2})^{\frac{1}{3}}$. The coefficient of $(dt+k)^2$ is W^{-4}, which vanishes as V^2. The singular part of the cross term, $dt\,k$, is $\mu\; dt\; (d\psi + A)$, which, from (92), diverges as V^{-2}, and so the overall cross term, $W^{-4} dt\; k$, remains finite at $V = 0$.

So the metric is regular at critical surfaces. The inverse metric is also regular at $V = 0$ because the $dt\; d\psi$ part of the metric remains finite, and so the determinant is non-vanishing.

This surface is therefore not an event horizon even though the time-like Killing vector defined by translations in t becomes null when $V = 0$. Indeed, when a metric is stationary but not static, the fact that g_{tt} vanishes on a surface does not make it an event horizon (the best known example of this is the boundary of the ergosphere of the Kerr metric). The necessary condition for a surface to be a horizon is rather to have $g^{rr} = 0$, where r is the coordinate transverse to this surface. This is clearly not the case here.

Hence, the surface given by $V = 0$ is like a boundary of an ergosphere, except that the solution has no ergosphere[18] because this Killing vector is time-like on both sides and does not change character across the critical surface. In the Kerr metric, the time-like Killing vector becomes space-like and this enables energy extraction by the Penrose process. Here there is no ergosphere and so energy extraction is not possible, as is to be expected from a BPS geometry.

At first sight, it does appear that the Maxwell fields are singular on the surface $V = 0$. Certainly the "magnetic components" $\Theta^{(I)}$ (see (77)) are singular when $V = 0$. However, one knows that the metric is non-singular, and so one should expect the singularity in the $\Theta^{(I)}$ to be unphysical. This intuition is correct: One must remember that the complete Maxwell fields are the $A^{(I)}$, and these are indeed non-singular at $V = 0$. One finds that the singularities in the "magnetic terms" of $A^{(I)}$ are canceled by singularities in the "electric terms" of $A^{(I)}$, and this is possible at $V = 0$ precisely because g_{tt} goes to zero, and so the magnetic and electric terms can communicate. Specifically, one has, from (36) and (81):

$$dA^{(I)} = d\left(B^{(I)} - \frac{(dt+k)}{Z_I}\right). \tag{107}$$

Near $V = 0$ the singular parts of this behave as:

$$\begin{aligned} dA^{(I)} &\sim d\left(\frac{K^I}{V} - \frac{\mu}{Z_I}\right)(d\psi + A) \\ &\sim d\left(\frac{K^I}{V} - \frac{K^1\, K^2\, K^3}{\frac{1}{2}\, V\, C_{IJK}\, K^J\, K^K}\right)(d\psi + A) \sim 0. \end{aligned} \tag{108}$$

[18] The non-supersymmetric smooth three-charge solutions found in [103] do nevertheless have ergospheres [103, 104].

The cancelations of the V^{-1} terms here occur for much the same reason that they do in the metric (100).

Therefore, even if V vanishes and changes sign and the base metric becomes negative definite, the complete eleven-dimensional solution is regular and well behaved around the $V = 0$ surfaces. It is this fact that gets us around the uniqueness theorems for asymptotically Euclidean self-dual (hyper-Kähler) metrics in four dimensions, and as we will see, there are now a vast number of candidates for the base metric.

5.5 *Black Rings in Taub-NUT*

Having analyzed the general form of solutions with a GH base, it is interesting to re-examine the black ring solution of Sect. 4 and rewrite it in the form discussed in Sect. 5.1 with a trivial GH base (with $V = \frac{1}{r}$). We do this because it is then elementary to generalize the solution to more complicated base spaces and most particularly to a Taub-NUT base. This will then illustrate a very important technique that makes it elementary to further compactify solutions to four-dimensional space-times and establish the relationship between four-dimensional and five-dimensional quantities. For pedagogical reasons, we will focus on the metric details on the field strengths and the moduli can be found in [84].

Exercise 8. *Show that the black ring warp factors and rotation vector, when written in usual $\mathbb{R}^4$ coordinates*

$$ds^2 = d\tilde{r}^2 + \tilde{r}^2\left(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\psi}^2 + \cos^2\tilde{\theta}d\tilde{\phi}^2\right) \tag{109}$$

are given by:

$$\begin{aligned} Z_I &= 1 + \frac{\overline{Q}_I}{\tilde{\Sigma}} + \frac{1}{2}C_{IJK}q^J q^K \frac{\tilde{r}^2}{\tilde{\Sigma}^2} \\ k &= -\frac{\tilde{r}^2}{2\tilde{\Sigma}^2}\left(q^I\overline{Q}_I + \frac{2q^1q^2q^3\tilde{r}^2}{\tilde{\Sigma}}\right)\left(\cos^2\tilde{\theta}\; d\tilde{\phi} + \sin^2\tilde{\theta}\; d\tilde{\psi}\right) \\ &\quad - J_T\;\frac{2\tilde{r}^2\sin^2\tilde{\theta}}{\tilde{\Sigma}(\tilde{r}^2 + \widetilde{R}^2 + \tilde{\Sigma})}\; d\tilde{\psi}, \end{aligned} \tag{110}$$

where $C_{IJK} = 1$ for $(IJK) = (123)$ and permutations thereof,

$$\tilde{\Sigma} \equiv \sqrt{(\tilde{r}^2 - \widetilde{R}^2)^2 + 4\widetilde{R}^2\tilde{r}^2\cos^2\tilde{\theta}}, \tag{111}$$

and $J_T \equiv J_{\tilde{\psi}} - J_{\tilde{\phi}}$.

In the foregoing, we have written the solution in terms of the ring charges, $\overline{Q}_I$, and, as we have already noted, for the five-dimensional black ring, these charges differ from the charges measured at infinity because of the charge "dissolved" in the M5-brane fluxes. The charges measured at infinity are $Q_I = \overline{Q}_I + \frac{1}{2}C_{IJK}q^J q^K$. We will also make a convenient choice of units in which $G_5 = \frac{\pi}{4}$, and choose the three T^2s of the M-theory metric to have equal size.

Exercise 9. *Show that in these units the charges Q_I, $\overline{Q}_I$, and q^I that appear in the supergravity warp factors are the same as the corresponding quantized brane charges.*

Hint 1: Begin by relating G_{11} and G_5 using the torus volumes.

Hint 2: You can cheat and use the relation between the charges in the supergravity formula and the integer quantized charges derived in [68] and summarized in (53). If you feel like doing honest character-building work, find the M2 charges by integrating F_7 over the corresponding $S^3 \times T^2 \times T^2$ at infinity; find the M5 charges by integrating F_4 over the corresponding $T^2 \times S^2$, where the S^2 goes around the ring.

Hint 3: You can find the M5 dipole charges most easily if you use a coordinate system centered at $\Sigma = 0$ described in (120).

From (65), the radius of the ring, $\widetilde{R}$, and is related to J_T by

$$J_T = (q^1 + q^2 + q^3)\widetilde{R}^2. \tag{112}$$

We now perform a change of coordinates, to bring the black ring to a form that can easily be generalized to Taub-NUT. Define

$$\phi = \tilde{\phi} - \tilde{\psi}, \quad \psi = 2\tilde{\psi}, \quad \theta = 2\tilde{\theta}, \quad \rho = \frac{\tilde{r}^2}{4}, \tag{113}$$

where the ranges of these coordinates are given by

$$\theta \in (0,\pi), \quad (\psi,\phi) \cong (\psi + 4\pi, \phi) \cong (\psi, \phi + 2\pi). \tag{114}$$

Exercise 10. *Verify that when $V = \frac{1}{\rho}$, the coordinate change (113) transforms the metric in the first line of (116) to that of flat $\mathbb{R}^4$.*

In the new coordinates, the black-ring metric is

$$\begin{aligned}
ds^2 &= -(Z_1Z_2Z_3)^{-2/3}(dt+k)^2 + (Z_1Z_2Z_3)^{1/3} h_{mn} dx^m dx^n, \\
Z_I &= 1 + \frac{\overline{Q}_I}{4\Sigma} + \frac{1}{2}C_{IJK}q^J q^K \frac{\rho}{4\Sigma^2}, \\
k &= \mu\,(d\psi + (1+\cos\theta)d\phi) + \omega, \\
\mu &= -\frac{1}{16}\frac{\rho}{\Sigma^2}\left(q^I\overline{Q}_I + \frac{2q^1q^2q^3\rho}{\Sigma}\right) + \frac{J_T}{16R}\left(1 - \frac{\rho}{\Sigma} - \frac{R}{\Sigma}\right), \\
\omega &= -\frac{J_T\rho}{4\Sigma(\rho+R+\Sigma)}\sin^2\theta\, d\phi,
\end{aligned} \tag{115}$$

with

$$h_{mn}dx^m dx^n = V^{-1}\left(d\psi + (1+\cos\theta)d\phi\right)^2 + V\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right),$$

$$V = \frac{1}{\rho}, \quad \Sigma = \sqrt{\rho^2 + R^2 + 2R\rho\cos\theta}, \quad R = \frac{\widetilde{R}^2}{4}. \tag{116}$$

Exercise 11. *Check that the solution (115) has the form described in Sect. 5.1 with the eight harmonic functions:*

$$K^I = -\frac{q^I}{2\Sigma}, \qquad L_I = 1 + \frac{\overline{Q}_I}{4\Sigma},$$

$$M = \frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{\Sigma}\right), \qquad V = \frac{1}{\rho}. \tag{117}$$

We should also note for completeness that the conventions we use here for these harmonic functions are those of [84] and differ from those of [79] by various factors of two. When $\mathbb{R}^4$ is written in Gibbons-Hawking form, the ring is sitting at a distance R along the negative z-axis of the three-dimensional base. Adding more sources on the z axis corresponds to making concentric black rings [79, 105].

Exercise 12. *Show that adding sources on the same side of the origin in the $\mathbb{R}^3$ base of (116), correspond to rings that sit in the same $\mathbb{R}^2$ inside $\mathbb{R}^4$. Show that rings that sit in orthogonal $\mathbb{R}^2$*s *inside $\mathbb{R}^4$ correspond to sources sitting on opposite sides of the origin of the $\mathbb{R}^3$ base of (116).*

To change the four-dimensional base metric into Taub-NUT one simply needs to add a constant, h, to the harmonic function V:

$$V = h + \frac{1}{\rho}. \tag{118}$$

Since the functions in the metric are harmonic, equations (88), (89), (92), (93), and (117), still imply that we have a supersymmetric solution. Actually, in order to avoid both Dirac string singularities and closed time-like curves, the relation (112) between J_T and the dipole charges must be modified to:

$$J_T\left(h + \frac{1}{R}\right) = 4(q^1 + q^2 + q^3). \tag{119}$$

This is discussed in detail in [82, 83, 84] and in later sections here, but it follows because the absence of singularities in ω puts constraints on the sources on the right-hand side of (93).

For small ring radius (or for small h), $R \ll h^{-1}$, this reduces to the five-dimensional black ring described earlier. We now wish to consider the opposite limit, $R \gg h^{-1}$. However, to keep "the same ring" we must keep all its quantized charges fixed, and so (119) means $h + \frac{1}{R}$ must remain constant. We can think of this as keeping the physical radius of the ring fixed while changing its position in Taub-NUT: The ring slides to a point where the physical ring radius is the same as the physical size of the compactification circle. In the limit where R is large, the black ring is far from the Taub-NUT center, and it is effectively wrapped around an infinite cylinder. In other words, it has become a straight black string wrapped on a circle and, from the four-dimensional perspective, it is point-like and is nothing but a four-dimensional black hole.

To see this in more detail, we consider the geometry in the region far from the tip, that is, for $\rho \gg 1$, where we can take $V = h$. We also want to center the three-dimensional spherical coordinates on the ring, and so we change to coordinates such that Σ is the radius away from the ring. We then have:

$$d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) = d\Sigma^2 + \Sigma^2 \left(d\hat{\theta}^2 + \sin^2\hat{\theta} d\phi^2\right), \tag{120}$$

and

$$\rho = \sqrt{\Sigma^2 + R^2 - 2R\Sigma\cos\hat{\theta}}, \quad \cos\theta = \frac{\Sigma\cos\hat{\theta} - R}{\rho}. \tag{121}$$

Taking $R \to \infty$, at fixed $(\Sigma, \hat{\theta}, \hat{\phi})$ and $h + \frac{1}{R}$, we find that the metric is:

$$\begin{aligned} ds^2 &= -\left(\tilde{Z}_1\tilde{Z}_2\tilde{Z}_3\right)^{-2/3}(d\tilde{t} + \tilde{\mu} d\psi)^2 \\ &\quad + \left(\tilde{Z}_1\tilde{Z}_2\tilde{Z}_3\right)^{1/3}\left(dr^2 + r^2(d\hat{\theta}^2 + \sin^2\hat{\theta} d\hat{\phi}^2) + d\psi^2\right), \end{aligned} \tag{122}$$

where

$$\tilde{Z}_I \equiv \frac{Z_I}{h}, \qquad \tilde{\mu} \equiv \frac{\mu}{h}, \qquad r \equiv h\Sigma, \qquad \tilde{t} \equiv \frac{t}{h}. \tag{123}$$

Note that the spatial section of (122) is precisely $\mathbb{R}^3 \times S^1$. When written in terms of the coordinate r the metric functions become:

$$\tilde{Z}_I = \frac{1}{h} + \frac{\overline{Q}_I}{4r} + \frac{C_{IJK} q^J q^K}{8r^2}, \qquad \tilde{\mu} = -\frac{J_T}{16r} - \frac{q^I \overline{Q}_I}{16r^2} - \frac{q^1 q^2 q^3}{8r^3}, \qquad \omega = 0. \tag{124}$$

This is precisely the four-dimensional black hole found by wrapping the black string solution of [71] on a circle.

As noted in [56], the entropy of the five-dimensional black ring takes a simple form in terms of the quartic invariant of $E_{7(7)}$:

$$S = 2\pi\sqrt{J_4}, \tag{125}$$

where J_4 is given by (97) with

$$\begin{aligned} &x_{12} = \overline{Q}_1, \quad x_{34} = \overline{Q}_2, \quad x_{56} = \overline{Q}_3, \quad x_{78} = 0, \\ &y^{12} = q^1, \quad y^{34} = q^2, \quad y^{56} = q^3, \quad y^{78} = J_T = J_{\tilde{\psi}} - J_{\tilde{\phi}}. \end{aligned} \tag{126}$$

Hence, the "tube angular momentum" J_T plays the role of another charge in the four-dimensional black hole picture. From the five-dimensional perspective, J_T is the difference of the two independent angular momenta and is given by (112). Upon compactification on the Taub-NUT circle, J_T represents the momentum around that circle and, as is very familiar in Kaluza-Klein (KK) reduction, a KK momentum becomes a conserved charge in the lower dimension.

It has long been known that maximal supergravity in four dimensions has $E_{7(7)}$ duality group and that the general entropy for the corresponding class of four-dimensional black holes can be expressed in terms of the quartic invariant [97]. The observation in [56] thus provided the first clue as to the relationship between five-dimensional black rings and four-dimensional black holes. We now examine this relationship in more detail.

5.6 Parameters, Charges and the "4D-5D" Connection

As we have seen, the ability to introduce a constant, h, into V as in (118) enables us to interpolate between configurations in five-dimensional space-time and configurations in four-dimensional space-time. For small h, the Taub-NUT circle is very large and the configuration behaves as if it were in a five-dimensional space-time while, for large h, the Taub-NUT circle is small and the configuration is effectively compactified. The first connection between a five-dimensional configuration and such a four-dimensional solution was made in [46], where the simple two-charge supertube [33] was put in Taub-NUT, and was related to a two-centered, four-dimensional configuration of the type previously analyzed in [98, 99, 100]. One can also consider a four-dimensional black hole that has a non-trivial KKM charge, and that sits at the center of Taub-NUT. When the KKM charge is one, this black hole also has two interpretations, both as a four-dimensional and as a five-dimensional black hole [81]. Since one can interpolate between the five-dimensional and the four-dimensional regimes by changing the moduli of the solution, one can give microscopic descriptions of black rings and black holes both from a four-dimensional perspective and from a five-dimensional perspective. This is called the "4D-5D" connection. This connection enables us to relate the parameters and charges appearing in the five-dimensional description of a system to those appearing in the four-dimensional description. We now examine this more closely, and we will encounter some important subtleties. To appreciate these, we need to recall some of the background behind the BPS black ring solutions.

One of the reasons that makes the BPS black ring solution so interesting is that it shows the failure of black-hole uniqueness in five dimensions. To be more specific, for the round ($U(1) \times U(1)$ invariant) BPS black ring solution there are only five conserved quantities: the two angular momenta, $J_1,\ J_2$, and the three electric charges, Q_I, *as measured from infinity*. However, these rings are determined by seven parameters: $\overline{Q}_I,\ q^I$, and J_T. We have seen how these parameters are related to details of the constituent branes and we have stressed, in particular, that the q^I are dipole charges that, a priori, are not conserved charges and so cannot be measured from infinity in five dimensions. As discussed in Sect. 3.4, the true conserved charges in five dimensions are non-trivial combinations of the fundamental "brane parameters," $\overline{Q}_I$, q^I, and J_T.

5.6.1 5D Dipole Charges and 4D Charges

In discussing the conserved charges of a system there is a very significant assumption about the structure of infinity. To determine the charges one integrates various field strengths and their duals on certain Gaussian surfaces. If one changes the structure of infinity, one can promote dipoles to conserved charges or lose conserved charges. One sees this very explicitly in the case of Taub-NUT space (116): by turning a constant piece in harmonic function V, one replaces the S^3 at infinity of $\mathbb{R}^4$ by an $S^2 \times S^1$. In particular, the "dipole" charges, q^I, of the five-dimensional black-ring become conserved magnetic charges in the Taub-NUT space. This is evident from the identification in (126) in which the x^{AB} and y^{AB}, respectively, represent conserved electric and magnetic charges measured on the Gaussian two-spheres at infinity in $\mathbb{R}^3$. More generally, from (95), we see that the leading behavior of each of the eight harmonic functions, K^I, L_I, M, and V yields a conserved charge in the Taub-NUT compactification.

In terms of the thermodynamics of black holes and black rings, the conserved charges measured at infinity are thermodynamic state functions of the system and the set of state functions depends upon the asymptotic geometry of the "box" in which we place the system. If a solution has free parameters that cannot be measured by the thermodynamic state functions then these parameters should be thought of as special properties of a particular microstate, or set of microstates, of the system. Thus, in a space-time that is asymptotic to flat $\mathbb{R}^{4,1}$, one cannot identify the microstates of a particular round black-ring solution by simply looking at the charges and angular momenta at infinity. Moreover, given a generic microstate with certain charges it is not possible to straightforwardly identify the black ring (or rings) to which this microstate corresponds. The only situation in which one can do this is when there exists a box in which one can place both the ring and the microstate, and one uses the box to define extra state functions that the two objects must share. Putting these objects in Taub-NUT and changing the moduli such that both the ring and the microstate have a four-dimensional interpretation allows one to define a box that can be used to measure the "specialized microstate structure" (i.e. the dipoles), as charges at infinity in four dimensions.

A good analogy is the thermodynamics and the kinetic theory of gases. The conserved charges correspond to the state functions while the internal, constituent brane parameters correspond to details of the motions of molecules in particular microstates. The state functions are non-trivial combinations of parameters of microstate but do not capture all the individual microstate parameters. If the box is a simple cube then there is no state function to capture vorticity, but there is such a state function for a toroidal box.

Thus solutions come with two classes of parameters: Those that are conserved and can be measured from infinity and those that represent particular, internal configurations of the thermodynamic system. There are two ways in which one can hope to give a microscopic interpretation of black rings. One is to take a near-horizon limit in which the black ring solution becomes asymptotically $AdS_3 \times S^3 \times T^4$ [56, 78] and to describe the ring in the D1-D5-P CFT dual to this system. The other is to focus on the near-ring geometry (or to put the ring in Taub-NUT) and describe it as a four-dimensional BPS black hole [56, 82, 84, 106], using the microscopic description of 4D black holes constructed given in [107, 108].

If one wants to describe black rings in the D1-D5-P CFT, it is, a priori, unclear how the dipole charges, which are not conserved charges (state functions) appear in this CFT. A phenomenological proposal for this has been put forth in [56], but clearly more work remains to be done. Moreover, the obvious partition functions that one can define and compute in this CFT [109, 110, 111], which only depend on the charges and angular momenta, cannot be compared to the bulk entropy of a particular black ring. One rather needs to find the ring (or rings) with the largest entropy for a given set of charges and match their entropy to that computed in the CFT.

Moreover, if one wants to describe the ring using a CFT corresponding to a four-dimensional black hole, it is essential to identify the correct M2 charges of the ring. The beauty of the brane description (or any other stringy description) of supertubes and black rings is that it naturally points out what these charges are.

5.6.2 5D Electric Charges and their 4D Interpretation

There has been some discussion in the literature about the correct identification of the charges of the black-ring system. In particular, there was the issue of whether the Q_I or the $\overline{Q}_I$ are the "correct" charges of the black-ring. There is no dispute about the charge measured at infinity, the only issue was the physical meaning, if any, to the $\overline{Q}_I$. In [112], it was argued that the only meaningful charge was the "Page charge" that measures Q_I and not $\overline{Q}_I$ even when the Gaussian surface is small surface surrounding the black ring. This is an interesting, mathematically self-consistent view, but it neglects a lot of the important underlying physics. It also generates some confusion as to the proper identification of the microscopic charges of the underlying system. The competing view [68] is the one we have presented here: The $\overline{Q}_I$ represent the number of constituent M2 branes and the Q_I get two contributions, one from the $\overline{Q}_I$ and another from the "charges dissolved in fluxes" arising from the M5 branes. It is certainly true that the $\overline{Q}_I$ and the q^I are not conserved individually, but they do represent critically important physical parameters.

This is easily understood in analogy with a heavy nucleus. The energy of the nucleus has two contributions, one coming from the rest mass of the neutrons and protons, and the other coming from the interactions between them. In trying to find the "microscopic" features of the nucleus, like the number of nucleons, one obtains an incorrect result, if one simply divides the total energy by the mass of a nucleon. To find the correct answer one should first subtract the energy coming from interactions and then divide the remainder by the mass of the nucleon.

One of the nice features of Taub-NUT compactification and the "4D-5D" connection is that it provides a very simple resolution of the foregoing issue in the identification of constituent microscopic charges. If one simply compactifies M-theory on $T^6 \times S^1$ from the outset, wrapping q^I M5 branes on the S^1 and each of the tori as shown in Table 1, then the q^I simply emerge as magnetic charges in four dimensions as in (126). Similarly, the $\overline{Q}_I$ are, unambiguously, the conserved electric charges of the system. This is also true of the Taub-NUT compactification of the black ring, and the fact that we can adiabatically vary h in (118) means we can bring the ring from a region that looks like M-theory on $T^6 \times S^1 \times \mathbb{R}^{3,1}$ into a region that looks like M-theory on $T^6 \times \mathbb{R}^{4,1}$ and still have confidence that the identification is correct because M2 and M5 brane charges are quantized and cannot jump in an adiabatic process. This establishes that the microscopic charges of the black ring are not the same as the charges measured at infinity in the five-dimensional black ring solution.

There are, of course, many situations where the rings cannot be put in Taub-NUT, and one cannot obtain the microscopic charges using the 4D-5D connection. The simplest example is the black ring with a black hole offset from its center [93] that we reviewed in Sect. 3.5. However, based upon our experience with the single black ring, we expect that the values of $\overline{Q}_I$ in the near-ring geometry will yield the number of M2 brane constituents of each individual ring.

There has also been a proposal to understand the entropy of BPS black rings in terms of microscopic charges, in which Q_I are interpreted as the M2 brane charges. This is based on a four-dimensional black hole CFT with charges Q_I rather than $\overline{Q}_I$, and with momentum J_ψ rather than J_T [106]. In order to recover the entropy formula (58), (59), an important role in that description was played by a non-extensive zero point energy shift of L_0. In light of our analysis, it is rather mysterious why this gives the right entropy, since we have shown explicitly that the relevant four-dimensional black hole CFT is the one with charges $\overline{Q}_I$, momentum J_T, and no zero point shift of L_0. We should also note that the approach of [106] seems to run into problems when describing concentric black rings because the total charge Q_A is not simply a sum of the individual $Q_{A,i}$ but gets contributions from cross terms of the form $C_{ABc}\, q_i^B\, q_j^C$. The approach of [106] also appears not to correctly incorporate some of the higher order corrections to the black ring entropy [113, 114].

One of the other benefits of the 4D-5D connection is that it also unites what have been two parallel threads in research. Prior to this there had been extensive and largely independent bodies of research on four-dimensional objects and upon on five-dimensional objects. It is now evident that the four-dimensional two-center solution corresponding to black rings and supertubes in Taub-NUT [46, 82, 83, 84] is part of the family of multi-center solutions that have been explored by Denef and collaborators [98, 99, 100]. In fact, one can also imagine putting in Taub-NUT multiple concentric black rings of the type studied by Gauntlett and Gutowski in [79, 105]. These descend in four dimensions to a multi-black hole configuration, in which the center of the rings becomes a center of KKM charge one, the black rings in one plane become black holes on the right of the KKM center, and the black rings in the other plane becomes black holes on the left of this center.

More generally, we expect that the 4D-5D connection will lead to a valuable symbiosis. For example, the work on attractor flows in Calabi-Yau manifolds and the branching of these flows could have important consequences for the bubbled geometries that we will discuss in the next section.

6 Bubbled Geometries

6.1 The Geometric Transition

The main purpose of our investigation thus far has been to construct smooth horizonless geometries starting from three-charge supertubes. We have seen that if one considers a process in which one takes a three-charge, three-dipole charge supertube to a regime where the gravitational back-reaction becomes important, the resulting supergravity solution is generically that of a BPS black ring. Although black rings are very interesting in their own right, they do have event horizons, and therefore do not correspond to microstates of the boundary theory.

Hence it is natural to try to obtain microstates by starting with brane configurations that do not develop a horizon at large effective coupling or alternatively to consider a black ring solution in the limit where its entropy decreases and becomes zero. However, the geometry of a zero-entropy black ring is singular. This singularity is not a curvature singularity, since the curvature is bounded above by the inverse of the dipole charges. Rather, the singularity is caused by the fact that the size of the S^1 of the horizon shrinks to zero size and the result is a "null orbifold." One can also think about this singularity as caused by the gravitational back-reaction of the branes that form the three-charge supertube, which causes the S^1 wrapped by these branes to shrink to zero size.

Fortunately, string theory is very good at solving this kind of singularities, and the mechanism by which it does is that of "geometric transition." To understand what a geometric transition is, consider a collection of branes wrapped on a certain cycle. At weak effective coupling, one can describe these branes by studying the open strings that live on them. One can also find the number of branes by integrating the corresponding flux over a "Gaussian" cycle dual to that wrapped by the branes. However, when one increases the coupling, the branes back-react on the geometry and shrink the cycle they wrap to zero size. At the same time, the "Gaussian cycle" becomes large and topologically non-trivial (see Fig. 5). The resulting geometry has a different topology, and *no brane sources*; the only information about the branes is now in the integral of the flux over the blown-up dual "Gaussian cycle." Hence, even if in the open-string (weakly coupled) description we had a configuration of branes, in the closed-string (large effective coupling) description these branes have disappeared and have been replaced by a non-trivial topology with flux.

Geometric transitions appear in many systems [86, 115, 116, 117]. A classic example of such a system are the brane models that break an $\mathcal{N}=2$ superconformal field theory down to an $\mathcal{N}=1$ supersymmetric field theory [86, 118]. Typically,

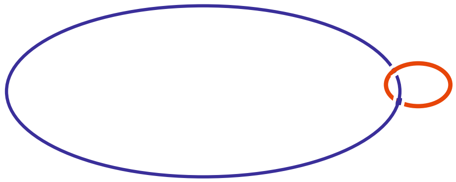

Fig. 5 *Geometric transitions*: The branes wrap the large cycle and the flux through the Gaussian (small, cycle measures the brane charge. In the open-string picture the small cycle has non-zero size, and the large cycle is contractible. After the geometric transition the size of the large cycle becomes zero, while the small cycle becomes topologically non-trivial

the $\mathcal{N} = 2$ superconformal field theory is realized on a stack of D3 branes in some Calabi-Yau compactification. One can then break the supersymmetry to $\mathcal{N} = 1$ by introducing extra D5 branes that wrap a two-cycle. When one investigates the closed-stringpicture, the two-cycle collapses and the dual three-cycle blows up (this is also known as a conifold transition). The D5 branes disappear and are replaced by non-trivial fluxes on the three-cycle. The resulting geometry has no more brane sources, and has a different topology than the one we started with.

Our purpose here is to see precisely how geometric transitions resolve the singularity of the zero-entropy black ring (*supertube*) of Sect. 3. Here the ring wraps a curve, $y^\mu(\sigma)$, that is topologically an S^1 inside $\mathbb{R}^4$ (in Fig. 6, this S^1 is depicted as a large, blue cycle). The Gaussian cycle for this S^1 is a two-sphere around the ring (illustrated by the red small cycle in Fig. 6). If one integrates the field strengths $\Theta^{(I)}$ on the red Gaussian two-cycle, one obtains the M5 brane dipole charges of the ring, n^I.

After the geometric transition the large (blue) S^1 becomes zero length, and the red S^2 becomes topologically non-trivial. Moreover, because the original topology is trivial, the curve $y^\mu(\sigma)$ was the boundary of a disk. When after the transition this boundary curve collapses, the disk becomes a (topologically non-trivial) two-sphere. Alternatively, one can think about this two-sphere (shown in Fig. 6 in green) as coming from having an S^1 that has zero size both at the origin of the space $r = 0$ and at the location of the ring. Hence, before the transition we had a ring wrapping

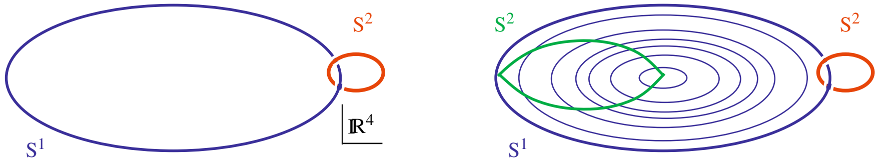

Fig. 6 *The geometric transition of the black ring*: Before the transition, the branes wrap the large (*blue*) S^1; the flux through the Gaussian S^2 (small, *red*) cycle measures the brane charge. After the transition the Gaussian S^2 (small, *red*) cycle is topologically non-trivial and of finite size and a new (*green*) S^2 appears, coming from the fact that the blue S^1 shrinks to zero so that the disk spanning the S^1 becomes an S^2. The resulting geometry has two non-trivial S^2s and no brane sources

a curve of arbitrary shape inside $\mathbb{R}^4$, and after the transition we have a manifold that is asymptotically $\mathbb{R}^4$, and has two non-trivial two-spheres, and no brane sources.

Can we determine the geometry of such a manifold? If the curve has an arbitrary shape the only information about this manifold is that it is asymptotically $\mathbb{R}^4$ and that it is hyper-Kähler, as required by supersymmetry.[19] If the curve wrapped by the supertube has arbitrary shape, this is not enough to determine the space that will come out after the geometric transition. However, if one considers a circular supertube, the solution before the transition has a $U(1) \times U(1)$ invariance, and so one naturally expects the solution resulting from the transition should also have this invariance.

With such a high level of symmetry, we do have enough information to determine what the result of the geometric transition is:

- By a theorem of Gibbons and Ruback [95], a hyper-Kähler manifold that has a $U(1) \times U(1)$ invariance must have a translational $U(1)$ invariance and hence must be Gibbons-Hawking.
- We also know that this manifold should have two non-trivial two-cycles, and hence, as discussed in Sect. 4.1 it should have three centers.
- Each of these centers must have integer GH charge.
- The sum of the three charges must be 1, in order for the manifold to be asymptotically $\mathbb{R}^4$.
- Moreover, we expect the geometric transition to be something that happens locally near the ring, and so we expect the region near the center of the ring (which is also the origin of our coordinate system) to remain the same. Hence, the GH center at the origin of the space must have charge $+1$.

The conclusion of this argument is that the space that results from the geometric transition of a $U(1) \times U(1)$ invariant supertube must be a GH space with three centers, that have charges 1, $+Q$, $-Q$, where Q is any integer. As we have seen in Sect. 5.5, equation (117), if we think about $\mathbb{R}^4$ as a trivial Gibbons-Hawking metric with $V = \frac{1}{r}$, the black ring solution of Sect. 3.4 has a GH center at the origin, and the ring at a certain point on the $\mathbb{R}^3$ base of the GH space. In the "transitioned" solution, the singularity of the zero-entropy black ring is resolved by the nucleation, or "pair creation," of two equal and oppositely charged GH points.

This process is depicted in Fig. 7. The nucleation of a GH pair of oppositely-charged centers blows up a pair of two-cycles. In the resolved geometry, there are no more brane sources, only fluxes through the two-cycles. The charge of the solution does not come from any brane sources but from having non-trivial fluxes over intersecting two-cycles (or "bubbles").

Similarly, if one considers the geometric transition of multiple concentric black rings, one will nucleate one pair of GH points for each ring, resulting in a geometry

[19] This might cause the faint-hearted to give up hope because of the theorem that the only such manifold is flat $\mathbb{R}^4$. This is the second instance when such theorems appear to preclude further progress in this research programme (the first is discussed at the end of Sect. 2). As in the previous example, we will proceed guided by the string-theory intuition, and will find a way to avoid the theorem.

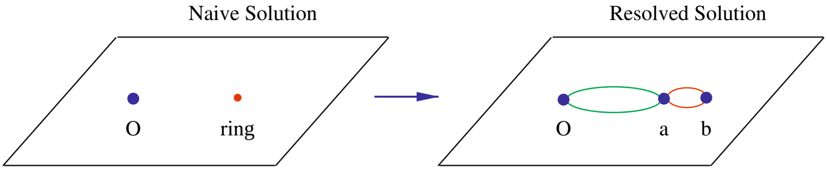

Fig. 7 *Geometric transition of supertube*: The first diagram shows the geometry before the transition. The second shows the resolved geometry, where a pair of GH charges has nucleated at positions *a* and *b*

with no brane sources, and with a very large number of positive and negative GH centers. As we will see, these centers are not restricted to be on a line but can have arbitrary positions in the $\mathbb{R}^3$ base of the GH space, as long as certain algebraic equations (discussed in Sect. 6.3) are satisfied.

There is one further piece of physical intuition that is extremely useful in understanding these bubbled geometries. As we have already remarked, GH points can be interpreted, from a ten-dimensional perspective, as D6 branes. Since these branes are mutually BPS, there should be no force between them. On the other hand, D6 branes of opposite charge attract one another, both gravitationally and electromagnetically. If one simply compactifies M-theory on an ambipolar GH space, one can only hold in equilibrium GH points of opposite charge at the cost of having large regions where the metric has the wrong signature and CTCs. To eliminate these singular regions, one must hold the GH points apart by some other mechanism. In the geometries we seek, this is done by having fluxes threading the bubbles: Collapsing a bubble concentrates the energy density of the flux and increases the energy in the flux sector. Thus a flux tends to blow up a cycle. The regular, ambipolar BPS configurations that we construct come about when these two competing effects – the tendency of oppositely charged GH points to attract each other and the tendency of the fluxes to make the bubbles large – are in balance. We will see precisely how this happens in Sect. 6.3.

Before proceeding to construct these solutions, we should note that there are two other completely different ways of arriving at the conclusion that three-charge black hole microstates can have a base given by a GH space with negative centers.

One direction, mostly followed by Mathur, Giusto, and Saxena [42, 43, 44] is to construct microstates by taking a novel extremal limit of the non-extremal five-dimensional black hole [119]. This limit produces a smooth horizonless geometries that have a GH base with two centers, of charges $N+1$ and $-N$. These geometries have a known CFT interpretation and form a subset of the class described above. A solution that is locally identical (but differs by a global identification of charges) was also found in [45] by doing a spectral flow on a two-charge solution and extending the resulting solution to an asymptotically flat one.

The second direction, followed by Kraus and one of the present authors is to consider the four-dimensional black hole with D1-D5-KKM-P charges, when the momentum is taken to zero. The resulting naive solution for the zero-entropy four-dimensional black hole is singular and is resolved by an intriguing mechanism: The branes that form the black hole split into two stacks, giving a non-singular solution [46]. One can then relate the black ring to a four-dimensional black hole by putting it in a Taub-NUT background, as discussed in Sect. 5.6 and in [82, 83, 84], and then the nucleation of a pair of oppositely charged GH centers corresponds, from a four-dimensional point of view, to the splitting of the zero-entropy four-dimensional black hole into two stacks of branes, giving a smooth resulting solution.

Hence, we have three completely independent routes for obtaining three-charge microstates and resolving the singularity of the zero-entropy black ring, and all three routes support the same conclusion: *The singularity of the zero-entropy black ring is resolved by the nucleation of GH centers of opposite charge. The solutions that result, as well as other three-charge microstate solutions, are topologically non-trivial, have no brane sources, and are smooth despite the fact that they are constructed using an ambipolar GH metric (with regions where the metric is negative-definite).*

6.2 The Bubbled Solutions

We now proceed to construct the general form of bubbling solutions constructed using an ambipolar Gibbons-Hawking base [47, 48, 49]. In Sect. 4.2, we saw that the two-forms, $\Theta^{(I)}$, will be *regular*, self-dual, harmonic two-forms, and thus representatives of the cohomology dual to the two-cycles, provided that the K^I have the form:

$$K^I = k_0^I + \sum_{j=1}^{N} \frac{k_j^I}{r_j}. \tag{127}$$

Moreover, from (87), the flux of the two-form, $\Theta^{(I)}$, through the two-cycle Δ_{ij} is given by

$$\Pi_{ij}^{(I)} = \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}\right), \qquad 1 \le i, j \le N. \tag{128}$$

The functions, L_I and M, must similarly be chosen to ensure that the warp factors, Z_I, and the function, μ, are regular as $r_j \to 0$. This means that we must take:

$$L^I = \ell_0^I + \sum_{j=1}^{N} \frac{\ell_j^I}{r_j}, \quad M = m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j}, \tag{129}$$

with

$$\ell_j^I = -\frac{1}{2}\, C_{IJK}\, \frac{k_j^J\, k_j^K}{q_j}, \qquad j = 1, \dots, N\,; \tag{130}$$

$$m_j = \frac{1}{12}\, C_{IJK} \frac{k_j^I\, k_j^J\, k_j^K}{q_j^2} = \frac{1}{2}\, \frac{k_j^1\, k_j^2\, k_j^3}{q_j^2}, \qquad j = 1, \dots, N. \tag{131}$$

Since we have now fixed the eight harmonic functions, all that remains is to solve for ω in (93). The right-hand side of (93) has two kinds of terms:

$$\frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} - \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i} \qquad \text{and} \qquad \vec{\nabla}\,\frac{1}{r_i}. \tag{132}$$

Hence ω will be built from the vectors $\vec{v}_i$ of (83) and some new vectors, $\vec{w}_{ij}$, defined by:

$$\vec{\nabla}\times\vec{w}_{ij} = \frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} - \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i}. \tag{133}$$

To find a simple expression for $\vec{w}_{ij}$, it is convenient to use the coordinates outlined earlier with the z-axis running through $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Indeed, choose coordinates so that $\vec{y}^{(i)} = (0,\,0,\,a)$ and $\vec{y}^{(j)} = (0,\,0,\,b)$ and one may take $a > b$. Then the explicit solutions may be written very simply:

$$w_{ij} = -\frac{(x^2+y^2+(z-a)(z-b))}{(a-b)\,r_i\,r_j}\,d\phi. \tag{134}$$

This is then easy to convert to a more general system of coordinates. One can then add up all the contributions to ω from all the pairs of points.

There is, however, a more convenient basis of vector fields that may be used instead of the w_{ij}. Define:

$$\omega_{ij} \equiv w_{ij} + \frac{1}{(a-b)}\,(v_i - v_j + d\phi) = -\frac{\big(x^2+y^2+(z-a+r_i)\,(z-b-r_j)\big)}{(a-b)\,r_i\,r_j}\,d\phi, \tag{135}$$

These vector fields then satisfy:

$$\vec{\nabla}\times\vec{\omega}_{ij} = \frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} - \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i} + \frac{1}{r_{ij}}\left(\vec{\nabla}\,\frac{1}{r_i} - \vec{\nabla}\,\frac{1}{r_j}\right), \tag{136}$$

where

$$r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}| \tag{137}$$

is the distance between the ith and jth center in the Gibbons-Hawking metric.

We then see that the general solution for $\vec{\omega}$ may be written as:

$$\vec{\omega} = \sum_{i,j}^{N} a_{ij}\,\vec{\omega}_{ij} + \sum_{i}^{N} b_i\,\vec{v}_i, \tag{138}$$

for some constants $a_{ij},\ b_i$.

The important point about the ω_{ij} is that they have *no string singularities whatsoever*. They can be used to solve (93) with the first set of source terms in (132), without introducing Dirac-Misner strings, but at the cost of adding new source terms of the form of the second term in (132). If there are N source points, $\vec{y}^{(j)}$, then using the w_{ij} suggests that there are $\frac{1}{2}N(N-1)$ possible string singularities associated with the axes between every pair of points $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. However, using the ω_{ij} makes it far more transparent that all the string singularities can be reduced to those associated with the second set of terms in (132), and so there are at most N possible string singularities, and these can be arranged to run in any direction from each of the points $\vec{y}^{(j)}$.

Finally, we note that the constant terms in (71), (127), and (129) determine the behavior of the solution at infinity. If the asymptotic geometry is Taub-NUT, all these terms can be non-zero, and they correspond to combinations of the moduli. However, in order to obtain solutions that are asymptotic to five-dimensional Minkowski space, $\mathbb{R}^{4,1}$, one must take $\varepsilon_0 = 0$ in (71), and $k_0^I = 0$ in (127). Moreover, μ must vanish at infinity, and this fixes m_0. For simplicity we also fix the asymptotic values of the moduli that give the size of the three T^2s, and take $Z_I \to 1$ as $r \to \infty$. Hence, the solutions that are asymptotic to five-dimensional Minkowski space have:

$$\varepsilon_0 = 0, \qquad k_0^I = 0, \qquad l_0^I = 1, \qquad m_0 = -\frac{1}{2}\, q_0^{-1} \sum_{j=1}^{N} \sum_{I=1}^{3} k_j^I. \tag{139}$$

It is straightforward to generalize these results to solutions with different asymptotics, and in particular to Taub-NUT.

6.3 The Bubble Equations

In Sect. 5.3, we examined the conditions for the absence of CTCs and in general the following must be true globally:

$$\mathcal{Q} \geq 0, \qquad V\, Z_I = \frac{1}{2}\, C_{IJK}\, K^J\, K^K + L_I\, V \geq 0, \qquad I = 1,2,3. \tag{140}$$

As yet, we do not know how to verify these conditions in general, but one can learn a great deal by studying the limits in which one approaches a Gibbons-Hawking point, *i.e.* $r_j \to 0$. From this, one can derive some simple, physical conditions (the *Bubble Equations*) that in some examples ensure that (140) are satisfied globally.

To study the limit in which $r_j \to 0$, it is simpler to use (98) than (100). In particular, as $r_j \to 0$, the functions, Z_I, μ, and W limit to finite values while V^{-1} vanishes. This means that the circle defined by ψ will be a CTC, unless we impose the additional condition:

$$\mu(\vec{y} = \vec{y}^{(j)}) = 0, \quad j = 1, \ldots, N. \tag{141}$$

There is also potentially another problem: The small circles in ϕ near $\theta = 0$ or $\theta = \pi$ will be CTCs if ω has a finite $d\phi$ component near $\theta = 0$ or $\theta = \pi$. Such a finite $d\phi$ component corresponds precisely to a Dirac-Misner string in the solution, and so we must make sure that ω has no such string singularities.

It turns out that these two sets of constraints are exactly the same. One can check this explicitly, but it is also rather easy to see from (90). The string singularities in

$\vec{\omega}$ potentially arise from the $\vec{\nabla}(r_j^{-1})$ terms on the right-hand side of (90). We have already arranged that the Z_I and μ go to finite limits at $r_j = 0$, and the same is automatically true of $K^I V^{-1}$. This means that the only term on the right-hand side of (90) that could, and indeed will, source a string is the $\mu\vec{\nabla}V$ term. Thus removing the string singularities is equivalent to (141).

One should note that by arranging that μ, ω, and Z_I are regular one has also guaranteed that the physical Maxwell fields, $dA^{(I)}$, in (107) are regular. Furthermore, by removing the Dirac strings in ω and by requiring μ to vanish at GH points, one has guaranteed that the physical flux of $dA^{(I)}$ through the cycle Δ_{ij} is still given by (87) (and (128)). This is because the extra terms, $d(Z_I^{-1}k)$, in (107), while canceling the singular behavior when $V = 0$, as in (108), give no further contribution in (87). Thus the fluxes, $\Pi_{ij}^{(I)}$, are well-defined and represent the true physical, magnetic flux in the five-dimensional extension of the ambipolar GH metrics.

Performing the expansion of μ using (92), (127), (129), and (131) around each Gibbons-Hawking point one finds that (141) becomes the *Bubble Equations*:

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{ij}^{(1)}\, \Pi_{ij}^{(2)}\, \Pi_{ij}^{(3)}\, \frac{q_i\, q_j}{r_{ij}} = -2 \left(m_0\, q_i + \frac{1}{2} \sum_{I=1}^{3} k_i^I \right), \tag{142}$$

where $r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}|$. One obtains the same set of equations, if one collects all the Dirac string contributions to ω and sets them to zero by imposing $b_i = 0$ in (138). If one adds together all of the bubble equations, then the left-hand side vanishes identically, and one obtains the condition on m_0 in (139). This is simply the condition $\mu \to 0$ as $r \to \infty$ and means that there is no Dirac-Misner string running out to infinity. Thus there are only $(N-1)$ independent bubble equations.

We refer to (142) as the bubble equations because they relate the flux through each bubble to the physical size of the bubble, represented by r_{ij}. Note that for a generic configuration, a bubble size can only be non-zero if and only if *all three* of the fluxes are non-zero. Thus the bubbling transition will only be generically possible for the three-charge system. We should also note that from a four-dimensional perspective these equations describe a collection of BPS stacks of branes, and are thus particular case of a collection of BPS black holes. Such configurations have been constructed in [98, 99, 100], and the equations that must be satisfied by the positions of the black holes are called "integrability equations" and reduce to the equations (142) when the charges are such that the five-dimensional solution is smooth.

6.4 The Asymptotic Charges

As in Sect. 3.3, one can obtain the electric charges and angular momenta of bubbled geometries by expanding Z_I and k at infinity. It is, however, more convenient to translate the asymptotics into the standard coordinates of the Gibbons-Hawking spaces. Thus, remembering that $r = \frac{1}{4}\rho^2$, one has

$$Z_I \sim 1 + \frac{Q_I}{4\,r} + \ldots, \qquad \rho \to \infty, \tag{143}$$

and from (88) one easily obtains

$$Q_I = -2\,C_{IJK} \sum_{j=1}^{N} q_j^{-1}\,\tilde{k}_j^J\,\tilde{k}_j^K, \tag{144}$$

where

$$\tilde{k}_j^I \equiv k_j^I - q_j\,N\,k_0^I, \qquad \text{and} \qquad k_0^I \equiv \frac{1}{N}\sum_{j=1}^{N} k_j^I. \tag{145}$$

Note that $\tilde{k}_j^I$ is gauge invariant under (79).

One can read off the angular momenta using an expansion like that of (42). However, it is easiest to re-cast this in terms of the Gibbons-Hawking coordinates. The flat GH metric (near infinity) has $V = \frac{1}{r}$ and making the change of variable $r = \frac{1}{4}\rho^2$, one obtains the metric in spherical polar coordinates:

$$ds_4^2 = d\rho^2 + \frac{1}{4}\,\rho^2\,\Big(d\theta^2 + \sin^2\theta\,d\phi^2 + (d\psi + \cos\theta\,d\phi)^2\Big)\,. \tag{146}$$

This can be mapped to the form of (41) via the change of variable:

$$u\,e^{i\theta_1} = \rho\cos\left(\frac{1}{2}\theta\right)\,e^{\frac{i}{2}(\psi+\phi)}, \qquad v\,e^{i\theta_2} = \rho\sin\left(\frac{1}{2}\theta\right)\,e^{\frac{i}{2}(\psi-\phi)}. \tag{147}$$

Using this in (42) one finds that

$$k \sim \frac{1}{4\,\rho^2}\,\big((J_1+J_2) + (J_1-J_2)\,\cos\theta\big)\,d\psi + \ldots. \tag{148}$$

Thus, one can get the angular momenta from the asymptotic expansion of $g_{t\psi}$, which is given by the coefficient of $d\psi$ in the expansion of k, which is proportional to μ. There are two types of such terms, the simple $\frac{1}{r}$ terms and the dipole terms arising from the expansion of $V^{-1}\,K^I$. Following [48], define the dipoles

$$\vec{D}_j \equiv \sum_I \tilde{k}_j^I\,\vec{y}^{(j)}, \qquad \vec{D} \equiv \sum_{j=1}^{N} \vec{D}_j. \tag{149}$$

and then the expansion of k takes the form (148) if one takes $\vec{D}$ to define the polar axis from which θ is measured. One then arrives at

$$J_R \equiv J_1 + J_2 = \frac{4}{3}\;C_{IJK}\sum_{j=1}^{N} q_j^{-2}\,\tilde{k}_j^I\,\tilde{k}_j^J\,\tilde{k}_j^K, \tag{150}$$

$$J_L \equiv J_1 - J_2 = 8\,\big|\vec{D}\big|. \tag{151}$$

While we have put modulus signs around $\vec{D}$ in (151), one should note that it does have a meaningful orientation, and so we will sometimes consider $\vec{J}_L = 8\vec{D}$.

One can use the bubble equations to obtain another, rather more intuitive expression for $J_1 - J_2$. One should first note that the right-hand side of the bubble equation, (142), may be written as $-\sum_I \tilde{k}^I_i$. Multiplying this by $\vec{y}^{(i)}$ and summing over i yields:

$$\begin{aligned}\vec{J}_L \equiv 8\,\vec{D} &= -\frac{4}{3}\,C_{IJK} \sum_{\substack{i,j=1\\ j\neq i}}^{N} \Pi^{(I)}_{ij}\,\Pi^{(J)}_{ij}\,\Pi^{(K)}_{ij}\,\frac{q_i\,q_j\,\vec{y}^{(i)}}{r_{ij}} \\ &= -\frac{2}{3}\,C_{IJK} \sum_{\substack{i,j=1\\ j\neq i}}^{N} q_i\,q_j\,\Pi^{(I)}_{ij}\,\Pi^{(J)}_{ij}\,\Pi^{(K)}_{ij}\,\frac{(\vec{y}^{(i)}-\vec{y}^{(j)})}{\left|\vec{y}^{(i)}-\vec{y}^{(j)}\right|}, \end{aligned} \tag{152}$$

where we have used the skew symmetry $\Pi_{ij} = -\Pi_{ji}$ to obtain the second identity. This result suggests that one should define an angular momentum flux vector associated with the ijth bubble:

$$\vec{J}_{L\,ij} \equiv -\frac{4}{3}\,q_i\,q_j\,C_{IJK}\,\Pi^{(I)}_{ij}\,\Pi^{(J)}_{ij}\,\Pi^{(K)}_{ij}\,\hat{y}_{ij}, \tag{153}$$

where $\hat{y}_{ij}$ are *unit* vectors,

$$\hat{y}_{ij} \equiv \frac{(\vec{y}^{(i)}-\vec{y}^{(j)})}{\left|\vec{y}^{(i)}-\vec{y}^{(j)}\right|}. \tag{154}$$

This means that the flux terms on the left-hand side of the bubble equation actually have a natural spatial direction, and once this is incorporated, it yields the contribution of the bubble to J_L.

6.5 *Comments on Closed Time-Like Curves and the Bubble Equations*

While the bubble equations are necessary to avoid CTCs near the Gibbons-Hawking points, they are not sufficient to guarantee the absence of CTCs globally. Indeed, there are non-trivial examples that satisfy the bubble equations and still have CTCs. On the other hand, there are quite a number of important physical examples in which the bubble equations *do* guarantee the absence of CTCs globally. For example, the simplest bubbled supertube will be discussed in Sect. 7.1, and it has been shown numerically in some examples that the bubble equations do indeed ensure the global absence of CTCs. Some more complex examples that are free of CTCs are described in Sect. 8.4. It is an open question as to how and when a bubbled configuration that satisfies (142) is globally free of CTCs. In this section, we will make some simple observations that suggest approaches to solving this problem.

First, we need to dispel a "myth" or, more precisely, give a correct statement of an often mis-stated theorem that in a BPS solution of extremal black holes, all the black holes must have electric charges of the same sign. The physical intuition is simple: If two BPS black holes have opposite charges, then they necessarily attract both gravitationally and electromagnetically and cannot be stationary, and this time dependence breaks the supersymmetry of the original BPS solutions. While this is physically correct, there is an implicit assumption that we are not going to allow physical solutions to have CTCs, changes in metric signature, or regions with complex metrics. A simple example is to make a solution with BMPV black holes given by the harmonic function:

$$Z_I = Z \equiv 1 + \frac{Q}{r_1} - \frac{Q}{r_2}. \tag{155}$$

We are not advocating that solutions like this, or ones with CTCs in general, should be taken as physically sensible. Nevertheless, this solution does satisfy all the equations of motion. The point we wish to make is that if one takes a completely standard, multi-centered BPS solution one can get lots of CTCs or imaginary metric coefficients, if one is sloppy about the relative signs of the distributed charges. For this reason, one cannot expect to take a bubbled geometry and randomly assign some flux parameters and expect to avoid CTCs even if one has satisfied the bubble equations. Indeed, a bubbled analogue of the BMPV configuration (155) could easily satisfy the bubble equations thereby avoiding CTCs near the Gibbons-Hawking points, only to have all sorts of pathology in between the two bubbled black holes.

There must therefore be some kind of positivity condition on the flux parameters. One suggestion might be to look at every subset, $\mathscr{S}$, of the Gibbons-Hawking points. To such a subset one can associate a contribution, $Q_I^{(\mathscr{S})}$, to the electric charges by summing (144) only over the subset, $\mathscr{S}$. One could then require that the $Q_I^{(\mathscr{S})}$ have the same sign for all choices of $\mathscr{S}$. This would exclude bubbled analogs of (155), but it might also be too stringent. It may be that one can tolerate a "mild failure" of the conditions on relative signs of electric charges if the Gibbons-Hawking points are all clustered; the danger might only occur in "classical limits" when some fluxes are very large so that the solution decomposes into two widely separated "blobs" of opposite charge.

Another natural, and possibly related condition is to remember that given N Gibbons-Hawking centers, the cycles are related to the root lattice of $SU(N)$ and the dual fluxes can be labeled by the weight lattice. In this language, the obvious positivity condition is to insist that the fluxes all lie in the positive Weyl chamber of the lattice. Moreover, when there are N_1 positive and N_2 negative GH points, it may be more appropriate to think in terms of the weight lattice of a super Lie algebra, $SU(N_1|N_2)$. In this context, one can rewrite $Z_I V$ in a rather more suggestive manner:

$$Z_I \, V = V - \frac{1}{4} \, C_{IJK} \sum_{i,j=1}^{N} \Pi_{ij}^{(J)} \, \Pi_{ij}^{(K)} \, \frac{q_i \, q_j}{r_i \, r_j}. \tag{156}$$

With suitable positivity conditions on the fluxes, one can arrange all the terms with $q_i q_j < 0$ to be positive. It is not clear why, in general, these terms dominate the terms with $q_i q_j > 0$, but one can certainly verify it in examples like the one below.

Consider the situation where all the flux parameters corresponding to a given $U(1)$ are equal and positive:

$$k_j^1 = a, \qquad k_j^2 = b, \qquad k_j^3 = c, \qquad a,b,c > 0, \quad j = 1,\dots,N. \tag{157}$$

Also suppose that $q_j = \pm 1$ and let $P_\pm$ be the subsets of integers, j, for which $\pm q_j > 0$. Define

$$V_\pm \equiv \sum_{j \in P_\pm} \frac{1}{r_j}. \tag{158}$$

Then one has

$$Z_I\, V = V + 4\, h_I\, V_+\, V_-, \tag{159}$$

where $h_1 \equiv bc,\ h_2 \equiv ac,\ h_3 \equiv ab$.

For this flux distribution the bubble equations reduce to:

$$8\, abc\, V_-(\vec{y} = \vec{y}^{(i)}) = (N-1)\,(a+b+c) \qquad \text{for all } i \in P_+, \tag{160}$$

$$8\, abc\, V_+(\vec{y} = \vec{y}^{(j)}) = (N+1)\,(a+b+c) \qquad \text{for all } j \in P_-. \tag{161}$$

Multiply the first of these equations by r_i^{-1} and sum, and multiply the second equation by r_j^{-1} and sum, and one obtains:

$$V_+ = \frac{8\, abc}{(N-1)\,(a+b+c)} \sum_{i \in P_+} \sum_{j \in P_-} \frac{1}{r_{ij}}\, \frac{1}{r_i}, \tag{162}$$

$$V_- = \frac{8\, abc}{(N+1)\,(a+b+c)} \sum_{i \in P_+} \sum_{j \in P_-} \frac{1}{r_{ij}}\, \frac{1}{r_j}. \tag{163}$$

Now note that $V = V_+ - V_-$ and use the foregoing expressions in (159) to obtain:

$$Z_I\, V = \sum_{i \in P_+} \sum_{j \in P_-} \frac{1}{r_i\, r_j\, r_{ij}} \left[\frac{8\, abc}{(a+b+c)} \left(\frac{r_j}{(N-1)} - \frac{r_i}{(N+1)} \right) + 4\, h_I\, r_{ij} \right]. \tag{164}$$

Since $a,b,c > 0$ and $N > 1$, one has

$$\frac{8\, abc}{(N+1)\,(a+b+c)} < 4\, h_I, \qquad I = 1,2,3, \tag{165}$$

and thus the positivity of $Z_I V$ follows trivially from the triangle inequality:

$$r_j + r_{ij} \geq r_i. \tag{166}$$

Note that it was relatively easy to prove positivity under the forgoing assumptions and that there was a lot of “wiggle room” in establishing the inequality. More formally, one can show that (164) is uniformly bounded below in a large compact region, and so one can allow some variation of the flux parameters with j and still preserve positivity. It would, of course, be very nice to know what the possible ranges of flux parameters are.

7 Microstates for Black Holes and Black Rings

Having explored the general way to construct smooth three-charge bubbling solutions that have charges and angular momenta of the same type as three-charge black holes and black rings, we now turn to exploring such solutions in greater generality. We will begin by describing several simple examples, like the simplest bubbled black ring, or a bubbled black hole made of several bubbles. We will find that when the number of bubbles is large, and the fluxes on them are generic, these solutions have the same relation between charges and angular momenta as the maximally spinning (zero-entropy) three-charge BPS black hole ($J^2 = N_1 N_5 N_p$). Moreover, when all the GH centers except one are in the same blob, and one GH center sits away from the blob, the solutions have the same charges, dipole charges and angular momenta as a zero-entropy, three-charge BPS black ring. Thus, to any zero-entropy black hole or any round three-charge supertube there corresponds a very large number of bubbled counterparts.

It is interesting to recall how the upper bound on the angular momentum is obtained for the BMPV black hole: One takes a solution with $J^2 < N_1 N_2 N_3$ and imagines spinning it faster. As this happens, the closed time-like curves (CTCs) inside the horizon get closer and closer to the horizon. When J^2 becomes larger than $N_1 N_2 N_3$, these CTCs sit outside the horizon, and the solution has to be discarded as unphysical. A similar story happens with the black ring. What is remarkable is that this relation between the charges and angular momentum, which came from studying the solution near the horizon of the black hole and black ring, also comes out from investigating horizonless solutions with a large number of bubbles and generic fluxes. The fact that this coincidence happens both for black holes, and for black rings (as well as for BPS black holes and rings in any $U(1)^N$ $\mathcal{N} = 2$, five-dimensional, ungauged supergravity) is indicative of a stronger connection between black holes and their bubbling counterparts.

Nevertheless, the fact that generic bubbling solutions correspond to zero-entropy black holes or to zero-entropy black rings means that we have only found a special corner of the microstate geometries. One might suspect, for example, that this feature comes from using a GH base space, and that obtaining microstates of positive-entropy black holes might be impossible unless one considers a more general base space. As we will see, this is not the case: We will be able to obtain microstates of black holes with $J^2 < N_1 N_2 N_3$ by merging together zero-entropy black hole microstates and zero-entropy black ring microstates[20].

[20] Obviously, the term "zero-entropy" applies to the black hole and black ring whose microstate geometries we discuss and *not* to the horizonless microstate geometries themselves. Such horizonless microstate geometries trivially have zero entropy.

As we have seen in Sect. 3.5, unlike the merger of two BPS black holes, which is always irreversible, the merger of a BPS black hole and a BPS black ring can be reversible or irreversible, depending on the charges of the two objects. We therefore expect the merger of microstates to result in an zero-entropy microstate or a positive-entropy black-hole microstate, depending on the charges of the merging microstates. Moreover, since the merger can be achieved in a Gibbons-Hawking base, we will obtain positive-entropy black-hole microstates that have a Gibbons-Hawking base. However, as we will see in the following sections, the merger process will be full of surprises.

We will find there is a huge qualitative difference between the behavior of the internal structure of microstates in "reversible" and "irreversible" mergers.[21] A "reversible" merger of an zero-entropy black-hole microstate and an zero-entropy black-ring microstate produces another zero-entropy black-hole microstate. For reversible mergers we find the bubbles corresponding to the ring simply join the bubbles corresponding to the black hole, and form a bigger bubbled structure.

In an "irreversible" merger, as the ring bubbles and the black hole bubbles get closer and closer, we find that the distances between the GH points that form the black hole foam and the black ring foam also decrease. As one approaches the merger point, all the sizes in the GH base scale down to zero while preserving their relative proportions. In the limit in which the merger occurs, the solutions have $J_1 = J_2 < \sqrt{Q_1 Q_2 Q_3}$, and all the points have scaled down to zero size on the base. Therefore, it naively looks like the configuration is singular; however, the full physical size of the bubbles also depends on the warp factors, and taking these into account one can show that the physical size of all the bubbles that form the black hole and the black ring remains the same. The fact that the GH points get closer and closer together implies that the throat of the solution becomes deeper and deeper and more and more similar to the throat of a BPS black hole (which is infinite).

7.1 The Simplest Bubbled Supertube

As we have discussed in Sect. 6.1, we expect the solution resulting from the geometric transition of a zero-entropy black ring to contain three GH centers, of charges $q_1 = 1$, $q_2 = -Q$ and $q_3 = +Q$. The integral of the flux on the Gaussian two-cycle bubbled at the position of the ring gives the dipole charges of the latter, d^I. It is useful to define another physical variables f^I, measuring the fluxes through the other two-cycle:

$$d^I \equiv 2\,\left(k_2^I + k_3^I\right), \qquad f_I \equiv 2\,k_1^I + \left(1 + \frac{1}{Q}\right)\,k_2^I + \left(1 - \frac{1}{Q}\right)\,k_3^I. \tag{167}$$

Note that d^I and f^I are invariant under (94).

[21] With an obvious abuse of terminology, we will refer to such solutions as "reversible" and "irreversible" mergers of microstates with the understanding that the notion of reversibility refers to the classical black-hole and black-ring solutions to which the microstates correspond.

The electric charges of the bubbled tube are:

$$Q_I = C_{IJK}\, d^J\, f^K, \tag{168}$$

and the angular momenta are:

$$J_1 = -\frac{(Q-1)}{12\,Q}\, C_{IJK}\, d^I\, d^J\, d^K + \frac{1}{2}\, C_{IJK}\, d^I\, d^J\, f^K, \tag{169}$$

$$J_2 = \frac{(Q-1)^2}{24\,Q^2}\, C_{IJK}\, d^I\, d^J\, d^K + \frac{1}{2}\, C_{IJK}\, f^I\, f^J\, d^K. \tag{170}$$

In particular, the angular momentum of the tube is:

$$\begin{aligned} J_T = J_2 - J_1 = \; & \frac{1}{2}\, C_{IJK}\, \left(f^I\, f^J\, d^K - d^I\, d^J\, f^K\right) \\ & + \left(\frac{3\,Q^2 - 4\,Q + 1}{24\,Q^2}\right)\, C_{IJK}\, d^I\, d^J\, d^K. \end{aligned} \tag{171}$$

When the size of the 2–3 bubble (between GH charges q_2 and q_3) is small, this configuration can be thought of as the resolution of the singularity of the zero-entropy supertube, and has the same charges, angular momenta, and size as the naive zero-entropy black ring solution. In the bubble equations, the size of the 2–3 bubble comes from a balance between the attraction of oppositely charged *GH* points, and the fluxes that have a lot of energy when the cycle they wrap becomes very small. Hence, both when Q is large and when d is much smaller than f the solution approaches the naive zero-entropy black ring solution

Exercise 13. *Verify that in the limit of large Q, as well as in the limit $d/f \to 0$ equations (169) and (170) match exactly the charges and angular momenta of a three-charge black ring of zero entropy.*

One can also estimate, in this limit, the distance from the small 2–3 bubble to the origin of space, and find that this distance asymptotes to the radius, R_T, of the un-bubbled black ring solution (as measured in the $\mathbb{R}^3$ metric of the GH base), given by

$$J_T = 4\, R_T\, (d^1 + d^2 + d^3). \tag{172}$$

7.2 Microstates of Many Bubbles

We now consider bubbled solutions that have a large number of localized centers and show that these solutions correspond to maximally spinning (zero-entropy) BMPV black holes, or to maximally spinning BPS black rings [51]. The ring microstates have a blob of GH centers of zero total charge with a single GH center away from the blob while the black hole microstates have all the centers in one blob of net GH charge one. We will see that this apparently small difference can very significantly influence the solution of the bubble equations.

7.2.1 A Black-Hole Blob

We first consider a configuration of N GH centers that lie is a single "blob" and take all these centers to have roughly the same flux parameters, to leading order in N. To argue that such a blob corresponds to a BMPV black hole, we first need to show that $J_1 = J_2$. If the overall configuration has three independent $\mathbb{Z}_2$ reflection symmetries then this is trivial because the $\vec{D}_j$ in (151) will then come in equal and opposite pairs, and so one has $J_L = 0$. More generally, for a "random" distribution[22] the vectors $\hat{y}_{ij}$ (defined in (154)) will point in "random" directions and so the $\vec{J}_{L\,ij}$ will generically cancel one another at leading order in N. The only way to generate a non-zero value of J_L is to bias the distribution such that there are more positive centers in one region and more negative ones in another. This is essentially what happens in the microstate solutions constructed and analyzed by [42, 43]. However, a single generic blob will have $J_1 - J_2$ small compared to $|J_1|$ and $|J_2|$.

To compute the other properties of such a blob, we will simplify things by taking $N = 2M+1$ to be odd and assume that $q_j = (-1)^{j+1}$. Using the gauge invariance, we can take all of k^I_i to be positive numbers, and we will assume that they have small variations about their mean value:

$$k^I_j = k^I_0 \, (1 + \mathcal{O}(1)), \tag{173}$$

where k^I_0 is defined in (145). The charges are given by:

$$\begin{aligned}
Q_I &= -2\, C_{IJK} \sum_j q_j^{-1} \, (k^J_j - q_j N k^J_0)\, (k^K_j - q_j N k^K_0) \\
&= -2\, C_{IJK} \left(\sum_j q_j^{-1} k^J_j k^K_j - N k^J_0 \sum_j k^K_j - N k^K_0 \sum_j k^J_j + N^2 k^J_0 k^K_0 \sum_j q_j \right) \\
&= 2\, C_{IJK} \left(N^2 k^J k^K - \sum_j k^J_j k^K_j q_j^{-1} \right) \\
&\approx 2\, C_{IJK} \big(N^2 + \mathcal{O}(1)\big)\, k^J_0 k^K_0
\end{aligned} \tag{174}$$

where we used (173) and the fact that $|q_i| = 1$ only in the last step. In the large N limit, for M theory on T^6 we have

$$Q_1 \approx 4N^2 k^2_0 k^3_0 + \mathcal{O}(1), \quad Q_2 \approx 4N^2 k^1_0 k^3_0 + \mathcal{O}(1), \quad Q_3 \approx 4N^2_0 k^1 k^3_0 + \mathcal{O}(1). \tag{175}$$

We can make a similar computation for the angular momenta:

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[22] Such a distribution must, of course, satisfy the bubble equations, (142), but this will still allow a sufficiently random distribution of GH points.

$$\begin{aligned} J_R &= \frac{4}{3}\, C_{IJK} \sum_j q_j^{-2}\, (k_j^I - q_j N k_0^I)\, (k_j^J - q_j N k_0^J)\, (k_j^K - q_j N k_0^K) \\ &= \frac{4}{3}\, C_{IJK} \bigg(\sum_j q_j^{-2}\, k_j^I k_j^J k_j^K - 3N k_0^I \sum_j q_j^{-1}\, k_j^J k_j^K \\ &\quad + 3N^2 k_0^I k_0^J \sum_j k_j^K - N^3 k_0^I k_0^J k_0^K \sum_j q_j \bigg) \\ &\approx \frac{4}{3}\, C_{IJK} \big(N - 3N + 3N^3 - N^3 + \mathscr{O}(N) \big)\, k_0^I k_0^J k_0^K , \end{aligned} \tag{176}$$

where we used the fact that, for a “well behaved” distribution of positive k_i^I with $|q_j| = 1$, one has:

$$\sum_i q_i^{-1} k_i^J k_i^K = \sum_i q_i k_i^J k_i^K \approx k_0^J k_0^K , \qquad \sum_i k_i^I k_i^J k_i^K \approx N k_0^I k_0^J k_0^K . \tag{177}$$

Therefore we simply have:

$$J_R \approx 16 N^3 k_0^1 k_0^2 k_0^3 + O(N). \tag{178}$$

Since $J_L \approx 0$ for a generic blob at large N, we therefore have at leading order:

$$J_1^2 \approx J_2^2 \approx \frac{1}{4}\, J_R^2 \approx Q_1 Q_2 Q_3 , \tag{179}$$

and so, in the large-N limit, these microstates always correspond to a maximally spinning BMPV black hole.

Exercise 14. *Show that at sub-leading order in N*

$$\frac{J_R^2}{4 Q_1 Q_2 Q_3} - 1 \sim O\left(\frac{1}{N^2}\right). \tag{180}$$

Interestingly enough, the value of J_R is slightly bigger than $\sqrt{4 Q_1 Q_2 Q_3}$. However, this is not a problem because in the classical limit this correction vanishes. Moreover, it is possible to argue that a classical black hole with zero horizon area will receive higher-order curvature corrections, which usually increase the horizon area; hence a zero-entropy configuration will have J_R slightly larger then the maximal classically allowed value, by an amount that vanishes in the large N (classical) limit.

7.2.2 A Supertube Blob

The next simplest configuration to consider is one in which one starts with the blob considered above and then moves a single GH point of charge $+1$ out to a very large distance from the blob. That is, one considers a blob of total GH charge zero with a single very distant point of GH-charge $+1$. Since one now has a strongly "biased" distribution of GH charges one should now expect $J_1 - J_2 \neq 0$.

Again we will assume N to be odd and take the GH charge distribution to be $q_j = (-1)^{j+1}$, with the distant charge being the Nth GH charge. The blob therefore has $\frac{1}{2}(N-1)$ points of GH charge $+1$ and $\frac{1}{2}(N-1)$ points of GH charge -1. When seen from far away one might expect this blob to resemble the three-point solution described above with $Q = \frac{1}{2}(N-1)$. We will show that this is exactly what happens in the large-N limit.

To have the Nth GH charge far from the blob means that all the two-cycles, $\Delta_{j\,N}$, must support a very large flux compared to the fluxes on the Δ_{ij} for $i, j < N$. To achieve this we therefore take:

$$k_j^I = a_0^I\,(1 + \mathscr{O}(1)), \quad j = 1, \ldots, N-1, \qquad k_N^I = -b_0^I\,N. \tag{181}$$

where

$$a_0^I \equiv \frac{1}{(N-1)} \sum_{j=1}^{N-1} k_j^I. \tag{182}$$

We also assume that a_0^I and b_0^I are of the same order. The fluxes of this configuration are then:

$$\Pi_{ij}^{(I)} = \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}\right), \tag{183}$$

$$\Pi_{i\,N}^{(I)} = -\Pi_{N\,i}^{(I)} = -\left(\frac{k_i^I}{q_i} + N\,b_0^I\right), \quad i, j = 1, \ldots, N-1.$$

For this configuration one has:

$$k_0^I = \frac{(N-1)}{N}\,a_0^I - b_0^I, \qquad \tilde{k}_N^I = -(N-1)\,a_0^I, \tag{184}$$

$$\tilde{k}_j^I = k_j^I + q_j\,(N\,b_0^I - (N-1)\,a_0^I), \quad j = 1, \ldots, N-1. \tag{185}$$

Motivated by the bubbling black ring of [51] and Sect. 7.1, define the physical parameters:

$$d^I \equiv 2\,(N-1)\,a_0^I, \qquad f^I \equiv (N-1)\,a_0^I - 2\,N\,b_0^I. \tag{186}$$

Keeping only the terms of leading order in N in (144) and (150), one finds:

$$Q_I = C_{IJK} d^J f^K, \tag{187}$$

$$J_1 + J_2 = \frac{1}{2}\,C_{IJK}(d^I d^J f^K + f^I f^J d^K) - \frac{1}{24}\,C_{IJK} d^I d^J d^K. \tag{188}$$

Since the Nth point is far from the blob, we can take $r_{iN} \approx r_0$ and then the Nth bubble equation reduces to:

$$\frac{1}{6}\,C_{IJK}\sum_{j=1}^{N-1}\left(\frac{k_j^I}{q_j}+N\,b_0^I\right)\left(\frac{k_j^J}{q_j}+N\,b_0^J\right)\left(\frac{k_j^K}{q_j}+N\,b_0^K\right)\frac{q_j}{r_0}=(N-1)\sum_I a_0^I. \tag{189}$$

To leading order in N this means that the distance from the blob to the Nth point, r_0, in the GH space is given by:

$$r_0 \approx \frac{1}{2}\,N^2\left[\sum_I a^I\right]^{-1} C_{IJK}\,a_0^I\,b_0^J\,b_0^K$$
$$= \frac{1}{32}\left[\sum_I d^I\right]^{-1} C_{IJK}\,d^I\,\left(2f^J-d^J\right)\,\left(2f^K-d^K\right). \tag{190}$$

Finally, considering the dipoles (149), it is evident that, to leading order in N, $\vec{D}$ is dominated by the contribution coming from the Nth point and that:

$$J_1-J_2=8\,|\vec{D}|=8\,N\left(\sum_I a_0^I\right)r_0=4\,N^3\,C_{IJK}\,a_0^I\,b_0^J\,b_0^K \tag{191}$$
$$=\frac{1}{8}\,C_{IJK}\,d^I\,\left(2f^J-d^J\right)\,\left(2f^K-d^K\right). \tag{192}$$

Exercise 15. *Verify that the angular momenta and the radius of this bubbling supertube ((187), (188), (190), and (192)) match those of the simplest bubbling supertube described in Section 7.1 and therefore match those of a zero-entropy black ring.*

Thus, the bubbling supertube of many centers also has *exactly* the same size, angular momenta, charges and dipole charges as a zero-entropy black ring and should be thought of a microstate of the later.

8 Mergers and Deep Microstates

As we have seen in Sect. 3.5, a merger of a zero-entropy black ring and a zero-entropy black hole can produce both a zero-entropy black hole (reversible merger) or a non-zero-entropy one (irreversible merger). We expect that in a similar fashion, the merger of the microstates of zero-entropy black holes and zero-entropy black rings should produce microstates of both zero-entropy and positive-entropy black holes. Since we have already constructed zero-entropy black-hole microstates, we will mainly focus on irreversible mergers and their physics. One can learn more about reversible mergers of microstates in Sect. 6 of [53].

Even though the original black-ring plus black-hole solution that describes the merger in Sect. 3.5 and [93] does not have a tri-holomorphic $U(1)$ symmetry (and thus the merger of the corresponding microstates cannot be done using a GH base), one can also study the merger of black rings and black holes by considering a $U(1)\times U(1)$ invariant solution describing a black ring with a black hole in the center. As the ring is made smaller and smaller by, for example, decreasing its angular momentum, it eventually merges into the black hole. At the point of merger, this solution is identical to the merger described in Sect. 3.5 with the black ring grazing the black-hole horizon. Hence this $U(1) \times U(1)$ invariant solution can be used to study mergers where the black ring grazes the black hole horizon at the point of merger. As we have seen in Sect. 3.5, all the reversible mergers and some of the irreversible mergers belong to this class.

In the previous section, we have seen how to create bubbled solutions corresponding to a zero-entropy black ring and maximally spinning black holes. The generic bubbled solutions with GH base have a $U(1)$ symmetry corresponding to $J_R \equiv J_1 + J_2$, and if the GH points all lie on an axis then the solution is $U(1) \times U(1)$ invariant. We can, therefore, study the merger of bubbled microstates by constructing $U(1) \times U(1)$ invariant bubbling solutions describing a black ring with a black hole in the center. By changing some of the flux parameters of the solution, one can decrease the radius of the bubbling black ring and merge it into the bubbling black hole to create a larger bubbling black hole.

In this section, we consider a bubbling black hole with a very large number of GH centers, sitting at the center of the simplest bubbled supertube, generated by a pair of GH points.[23] We expect two different classes of merger solution depending upon whether the flux parameters on the bubbled black hole and bubbled black ring are parallel or not. These correspond to reversible and irreversible mergers, respectively. The reversible mergers involve the GH points approaching and joining the black-hole blob to make a similar, slightly larger black-hole blob [53]. The irreversible merger is qualitatively very different and we will examine it in detail. First, however, we will establish some general results about the charges and angular momenta of the bubbled solutions that describe a bubbled black ring of two GH centers with a bubbling black-hole at the center.

8.1 Some Exact Results

We begin by seeing what may be deduced with no approximations whatsoever. Our purpose here is to separate all the algebraic formulae for charges and angular momenta into those associated with the black hole foam and those associated with the bubbled supertube. We will consider a system of N GH points in which the first $N-2$ points will be considered to be a blob and the last two points will have $q_{N-1} = -Q$ and $q_N = Q$. The latter two points can then be used to define a bubbled black ring.

Let $\hat{k}_0^I$ denote the average of the flux parameters over the first $(N-2)$ points:

$$\hat{k}_0^I \equiv \frac{1}{(N-2)} \sum_{j=1}^{N-2} k_j^I, \tag{193}$$

[23] Of course it is straightforward to generalize our analysis to the situation where both the supertube and the black hole have a large number of GH centers. However, the analysis is simpler and the numerical stability is better for mergers in which the supertube is composed of only two points, and we have therefore focused on this.

and introduce k-charges that have a vanishing average over the first $(N-2)$ points:

$$\hat{k}^I_j \equiv k^I_j - (N-2)\, q_j\, \hat{k}^I_0, \qquad j = 1, \ldots, N-2. \tag{194}$$

We also parameterize the last two k^I-charges in exactly the same manner as for the bubbled supertube (see (167)):

$$\begin{aligned} d^I &\equiv 2\, \left(k^I_{N-1} + k^I_N\right), \\ f^I &\equiv 2\,(N-2)\, \hat{k}^I_0 + \left(1 + \frac{1}{Q}\right) k^I_{N-1} + \left(1 - \frac{1}{Q}\right) k^I_N. \end{aligned} \tag{195}$$

One can easily show that the charge (144) decomposes into

$$Q_I = \widehat{Q}_I + C_{IJK}\, d^J\, f^K, \tag{196}$$

where

$$\widehat{Q}_I \equiv -2\, C_{IJK} \sum_{j=1}^{N-2} q_j^{-1}\, \hat{k}^J_j\, \hat{k}^K_j. \tag{197}$$

The $\widehat{Q}_I$ are simply the charges of the black-hole blob, made of the first $(N-2)$ points. The second term in (196) is exactly the expression, (168), for the charges of a bubbled supertube with GH centers of charges $+1$, $-Q$ and Q and k-charges $(N-2)\hat{k}^I_0$, k^I_{N-1} and k^I_N, respectively. Thus the charge of this configuration decomposes exactly as if it were a black-hole blob of $(N-2)$ centers and a bubbled supertube.

There is a similar result for the angular momentum, J_R. One can easily show that:

$$J_R = \widehat{J}_R + d^I\, \widehat{Q}_I + j_R, \tag{198}$$

where

$$\hat{J}_R \equiv \frac{4}{3}\;C_{IJK} \sum_{j=1}^{N-2} q_j^{-2}\, \hat{k}^I_j\, \hat{k}^J_j\, \hat{k}^K_j, \tag{199}$$

and

$$j_R \equiv \frac{1}{2}\, C_{IJK}\, \left(f^I f^J d^K + f^I d^J d^K\right) - \frac{1}{24}\, \left(1 - Q^{-2}\right)\, C_{IJK}\, d^I d^J d^K. \tag{200}$$

The term, $\hat{J}_R$, is simply the right-handed angular momentum of the black-hole blob made from $N-2$ points. The "ring" contribution to the angular momentum, j_R, agrees precisely with $J_1 + J_2$ given by (169) and (170) for an isolated bubbled supertube. The cross term, $d^I \widehat{Q}_I$ represents the interaction of the flux of the bubbled ring and the charge of the black-hole blob. This interaction term is exactly the same as that found in Sect. 3.5 and in [68, 79, 93] for a concentric black hole and black ring.

Thus, as far as the charges and J_R are concerned, the complete system is behaving as though it were a black-hole blob of $(N-2)$ points interacting with a bubbled black ring defined by the points with GH charges $\pm Q$ and a single point with GH charge $+1$ replacing the black-hole blob. Note that no approximations were made in the foregoing computations, and the results are true independent of the locations of the GH centers.

To make further progress we need to make some assumptions about the configuration of the points. Suppose, for the moment, that all the GH charges lie on the z-axis at points z_i with $z_i < z_{i+1}$. In particular, the GH charges, $-Q$ and $+Q$, are located at z_{N-1} and z_N, respectively.

With this ordering of the GH points, the expression for $\vec{J}_L$ collapses to:

$$J_L = \frac{4}{3}\, C_{IJK} \sum_{1\le i<j\le N} q_i\, q_j\, \Pi_{ij}^{(I)}\, \Pi_{ij}^{(J)}\, \Pi_{ij}^{(K)}. \tag{201}$$

This expression can then be separated, just as we did for J_R, into a black-hole blob component, a ring component, and interaction cross-terms. To that end, define the left-handed angular momentum of the blob to be:

$$\widehat{J}_L = \frac{4}{3}\, C_{IJK} \sum_{1\le i<j\le N-2} q_i\, q_j\, \Pi_{ij}^{(I)}\, \Pi_{ij}^{(J)}\, \Pi_{ij}^{(K)}. \tag{202}$$

Note that

$$\Pi_{ij}^{(I)} \equiv \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}\right) = \left(\frac{\hat{k}_j^I}{q_j} - \frac{\hat{k}_i^I}{q_i}\right), \qquad 1 \le i, j \le N-2, \tag{203}$$

and so this only depends upon the fluxes in the blob.

The remaining terms in (201) may then be written in terms of $\hat{k}_j^I$, d^I, and f^I defined in (194) and (195). In particular, there are terms that depend only upon d^I and f^I, and then there are terms that are linear, quadratic, and cubic in $\hat{k}_j^I$ (and depend upon d^I and f^I). The linear terms vanish because $\sum_{j=1}^{N-2} \hat{k}_j^I \equiv 0$, the quadratic terms assemble into $\widehat{Q}_I$ of (197), and the cubic terms assemble into $\hat{J}_R$ of (199). The terms proportional to (199) cancel between the terms with $j = N-1$ and $j = N$, and one is left with:

$$J_L = \widehat{J}_L - d^I\, \widehat{Q}_I + j_L, \tag{204}$$

where j_L is precisely the angular momentum, J_T, of the tube:

$$j_L \equiv \frac{1}{2}\, C_{IJK}\, \left(d^I f^J f^K - f^I d^J d^K\right) + \left(\frac{3\,Q^2 - 4\,Q + 1}{24\,Q^2}\right) C_{IJK}\, d^I d^J d^K. \tag{205}$$

Observe that (200) and (205) are exactly the angular momenta of a simple bubbled ring, (169) and (170). Again, we see the cross-term from the interaction of the ring dipoles and the electric charge of the blob. Indeed, combining (198) and (204), we obtain:

$$J_1 = \widehat{J}_1 + j_1, \qquad J_2 = \widehat{J}_2 + j_2 - d^I\, \widehat{Q}_I, \tag{206}$$

which is exactly how the angular momenta of the classical ring-hole solution in Sect. 3.5 decomposed. In particular, the term coming from the interaction of the ring dipole moment with the black hole charge only contributes to J_2.

Exercise 16. *Check the decompositions (197), (199), and (204).*

The results obtained above are independent of whether the blob of $N-2$ points is a BMPV black-hole blob, or a more generic configuration. However, to study mergers we now take the blob to be a black-hole microstate, with $\widehat{J}_L = 0$. The end result of the merger process is also a BMPV black hole microstate, and so $J_L = 0$. Therefore, the *exact* merger condition is simply:

$$\Omega \equiv \frac{1}{2}\, C_{IJK}\, \left(d^I f^J f^K - f^I d^J d^K\right) + \left(\frac{3\,Q^2 - 4\,Q + 1}{24\,Q^2}\right) C_{IJK}\, d^I d^J d^K - d^I\, \widehat{Q}_I \\ = 0. \qquad (207)$$

Using (171), this may be written:

$$J_T - d^I\, \widehat{Q}_I = 0, \qquad (208)$$

which is precisely the condition obtained in Sect. 3.5 and [93] for a classical black ring to merge with a black hole at its equator.

One should note that the argument that led to the expressions (204) and (205) for J_L, and to the exact merger condition, (207), apply far more generally. In particular, we only needed the fact that the unit vectors, $\hat{y}_{ij}$, defined in (154), are all parallel for $j = N-1$ and $j = N$. This is approximately true in many contexts, and most particularly if the line between the $(N-1)$th and Nth points runs through the blob and the width of the blob is small compared to the distance to the two exceptional points.

One should also not be surprised by the generality of the result in (205). The angular momentum, J_T, is an intrinsic property of a black ring, and hence for a zero-entropy black ring, J_T can only depend on the ds and fs and cannot depend on the black hole charges (that is, the $\hat{k}^I_j$). Therefore, we could have obtained (205) by simply setting the black hole charge to zero, and then reading off J_T from the bubbling black ring solution of Sect. 7.1. Hence, one should think about the expression of J_T in (171) as a universal relation between intrinsic properties of the bubbled ring: J_T, d^I, and f^I.

8.2 Some Simple Approximations

We now return to a general distribution of GH points, but we will assume that the two "black ring points" (the $(N-1)$th and Nth points) are close together but very far from the black-hole blob of the remaining $(N-2)$ points. Set up coordinates in the geometric center of the black-hole blob, *i.e*. choose the origin so that

$$\sum_{i=1}^{N-2} \vec{r}_i = 0. \qquad (209)$$

Let $r_0 \equiv |\vec{r}_{N-1}|$ be the distance from the geometric center of the blob to the first exceptional point, and let $\hat{r}_0$ be the the unit vector in that direction. The vector, $\vec{\Delta} \equiv \vec{r}_N - \vec{r}_{N-1}$, defines the width of the ring. We will assume that the magnitudes $\Delta \equiv |\vec{\Delta}|$ and $r_j \equiv |\vec{r}_j|$ are small compared to r_0. We will also need the first terms of the multipole expansions:

$$\frac{1}{|\vec{r}_{N-1} - \vec{r}_j|} = \frac{1}{r_0} + \frac{\vec{r}_j \cdot \hat{r}_0}{r_0^2} + \dots \tag{210}$$

$$\frac{1}{|\vec{r}_N - \vec{r}_j|} = \frac{1}{r_0} + \frac{(\vec{r}_j - \vec{\Delta}) \cdot \hat{r}_0}{r_0^2} + \dots . \tag{211}$$

For simplicity, we will also assume that the two black-ring points (we will also call these points "exceptional points") are co-linear with the origin so that

$$r_N \equiv |\vec{r}_N| = r_0 + \Delta. \tag{212}$$

The last two bubble equations are then:

$$\frac{\gamma}{\Delta} - \sum_{j=1}^{N-2} \frac{q_j\, \alpha_j}{|\vec{r}_N - \vec{r}_j|} = \sum_I \left(N\, Q\, k_0^I - k_N^I\right), \tag{213}$$

$$-\frac{\gamma}{\Delta} + \sum_{j=1}^{N-2} \frac{q_j\, \beta_j}{|\vec{r}_{N-1} - \vec{r}_j|} = -\sum_I \left(N\, Q\, k_0^I + k_{N-1}^I\right) \tag{214}$$

where k_0^I is given in (145) and

$$\alpha_j \equiv \frac{1}{6}\, Q\, C_{IJK}\, \Pi^{(I)}_{j\,N}\, \Pi^{(J)}_{j\,N}\, \Pi^{(K)}_{j\,N} \tag{215}$$

$$= \frac{1}{6}\, Q\, C_{IJK} \left(\frac{k_N^I}{Q} - \frac{k_j^I}{q_j}\right) \left(\frac{k_N^J}{Q} - \frac{k_j^J}{q_j}\right) \left(\frac{k_N^K}{Q} - \frac{k_j^K}{q_j}\right), \tag{216}$$

$$\beta_j \equiv \frac{1}{6}\, Q\, C_{IJK}\, \Pi^{(I)}_{j\,(N-1)}\, \Pi^{(J)}_{j\,(N-1)}\, \Pi^{(K)}_{j\,(N-1)} \tag{217}$$

$$= -\frac{1}{6}\, Q\, C_{IJK} \left(\frac{k_{N-1}^I}{Q} + \frac{k_j^I}{q_j}\right) \left(\frac{k_{N-1}^J}{Q} + \frac{k_j^J}{q_j}\right) \left(\frac{k_{N-1}^K}{Q} + \frac{k_j^K}{q_j}\right), \tag{218}$$

$$\gamma \equiv \frac{1}{6}\, Q^2\, C_{IJK}\, \Pi^{(I)}_{(N-1)\,N}\, \Pi^{(J)}_{(N-1)\,N}\, \Pi^{(K)}_{(N-1)\,N} = \frac{1}{48}\, Q^{-1}\, C_{IJK}\, d^I\, d^J\, d^K. \tag{219}$$

It is also convenient to introduce

$$\alpha_0 \equiv \sum_{j=1}^{N-2} q_j\, \alpha_j, \qquad \beta_0 \equiv \sum_{j=1}^{N-2} q_j\, \beta_j. \tag{220}$$

If one adds (213) and (214) then the terms involving γ cancel and using (211) one then obtains:

$$\sum_{j=1}^{N-2} q_j \left[\alpha_j \left(\frac{1}{r_0} + \frac{(\vec{r}_j - \vec{\Delta}) \cdot \hat{r}_0}{r_0^2} \right) - \beta_j \left(\frac{1}{r_0} + \frac{\vec{r}_j \cdot \hat{r}_0}{r_0^2} \right) \right] = \frac{1}{2} \sum_I d^I. \tag{221}$$

One now needs to perform the expansions with some care. Introduce the flux vector:

$$X^I \equiv 2\, f^I - d^I - 4\,(N-2)\, \hat{k}_0^I, \tag{222}$$

and note that the fluxes between the blob and ring points are given by:

$$\Pi^{(I)}_{j\,(N-1)} = -\frac{1}{4}\, [\, X^I + Q^{-1}\, d^I + 4\, q_j^{-1}\, k_j^I \,], \tag{223}$$

$$\Pi^{(I)}_{j\,N} = -\frac{1}{4}\, [\, X^I - Q^{-1}\, d^I + 4\, q_j^{-1}\, k_j^I \,]. \tag{224}$$

In particular, the difference of these fluxes is simply the flux through the two-cycle running between the two ring points:

$$\Pi^{(I)}_{j\,N} - \Pi^{(I)}_{j\,(N-1)} = \frac{d^I}{2\,Q} = \Pi^{(I)}_{(N-1)\,N}. \tag{225}$$

For the ring to be far from the black hole, the fluxes $\Pi^{(I)}_{j\,(N-1)}$ and $\Pi^{(I)}_{j\,N}$ must be large. For the ring to be thin $(\Delta \ll r_0)$, these fluxes must be of similar order, or $\Pi^{(I)}_{(N-1)\,N}$ should be small. Hence we are assuming that $\frac{d^I}{2Q}$ is small compared to X^I. We are also going to want the black hole and the black ring to have similar charges and angular momenta, J_R, and one of the ways of achieving this is to make f^I, d^I, and $N\hat{k}_0^I$ of roughly the same order.

Given this, the leading order terms in (221) become:

$$\sum_{j=1}^{N-2} q_j \left[\frac{(\alpha_j - \beta_j)}{r_0} - \alpha_j\, \frac{\Delta}{r_0^2} \right] = \frac{1}{2} \sum_I d^I. \tag{226}$$

One can then determine the ring width, Δ, using (213) or (214). In particular, when the ring width is small while the ring radius is large, the left-hand side of each of these equations is the difference of two very large numbers of similar magnitude. To leading order we may therefore neglect the right-hand sides and use the leading monopole term to obtain:

$$\beta_0\, \frac{\Delta}{r_0} \approx \alpha_0\, \frac{\Delta}{r_0} = \left[\sum_{j=1}^{N-2} q_j\, \alpha_j \right] \frac{\Delta}{r_0} \approx \gamma, \tag{227}$$

and hence (221) becomes:

$$-\gamma+\sum_{j=1}^{N-2} q_j\,(\alpha_j-\beta_j)\approx\left[\frac{1}{2}\sum_I d^I\right] r_0. \tag{228}$$

Using the explicit expressions for α_j, β_j, and γ, one then finds:

$$\begin{aligned} r_0 \approx \quad & \left[4\sum_I d^I\right]^{-1}\left[\frac{1}{2}C_{IJK}\,\left(d^I f^J f^K - f^I d^J d^K\right)\right. \\ & \left. +\left(\frac{3\,Q^2-4\,Q+1}{24\,Q^2}\right) C_{IJK}\,d^I d^J d^K - d^I \widehat{Q}_I\right]. \end{aligned} \tag{229}$$

This is exactly the same as the formula for the tube radius that one obtains from (172) and (171). Note also that we have:

$$r_0\approx\left[4\sum_I d^I\right]^{-1}\left[\,j_L-d^I\widehat{Q}_I\,\right], \tag{230}$$

where j_L the angular momentum of the supertube (204). In making the comparison to the results of Sect. 3.5 recall that for a black ring with a black hole exactly in the center, the embedding radius in the standard, flat $\mathbb{R}^4$ metric is given by:

$$R^2=\frac{l_p^6}{L^4}\,\left[\sum d^I\right]^{-1}\,\left(J_T-d^I\widehat{Q}_I\right). \tag{231}$$

The transformation between a flat $\mathbb{R}^4$ and the GH metric with $V=\frac{1}{r}$ involves setting $r=\frac{1}{4}\rho^2$, and this leads to the relation $R^2=4R_T$. We therefore have complete consistency with the classical merger result.

Note that the classical merger condition is simply $r_0\to 0$, which is, of course, very natural. This might, at first, seem to fall outside the validity of our approximation; however we will see in the next section that for irreversible mergers one does indeed maintain Δ, $r_j\ll r_0$ in the limit $r_0\to 0$. Reversible mergers cannot however be described in this approximation and have to be analyzed numerically.

8.3 Irreversible Mergers and Scaling Solutions

All the results we have obtained in Sect. 8.1 and 8.2 apply equally to reversible and irreversible mergers. However, since our main purpose is to obtain microstates of a BPS back hole with classically large horizon area, we now focus on irreversible mergers.

We will show that an irreversible merger occurs in such a manner that the ring radius, r_0, the ring width, Δ, and a typical separation of points within the black-hole blob all limit to zero while their ratios all limit to finite values. We will call these scaling solutions, or scaling mergers. As the merger progresses, the throat of the solution becomes deeper and deeper, and corresponding redshift becomes larger and larger. The resulting microstates have a very deep throat and will be called "deep microstates."

Using the solution constructed in the previous sections, we begin decreasing the radius of the bubbled ring, r_0, by decreasing some of its flux parameters. We take all the flux parameters of the $(N-2)$ points in the blob to be parallel:

$$k_j^I = \hat{k}_0^I = k^I, \qquad j = 1, \ldots, N-2, \tag{232}$$

Further assume that all the GH charges in the black-hole blob obey $q_j = (-1)^{j+1}$, $j = 1, \ldots, N-2$. We therefore have

$$\begin{aligned} \widehat{Q}_I &= 2\,(N-1)(N-3)\, C_{IJK}\, k^J k^K, \\ \widehat{J}_R &= \frac{8}{3}\,(N-1)(N-2)(N-3)\, C_{IJK}\, k^I k^J k^K. \end{aligned} \tag{233}$$

Define:

$$\mu_i \equiv \frac{1}{6}\,(N-2-q_i)^{-1}\, C_{IJK} \sum_{\substack{j=1 \\ j\neq i}}^{N-2} \Pi_{ij}^{(I)}\, \Pi_{ij}^{(J)}\, \Pi_{ij}^{(K)}\, \frac{q_j}{r_{ij}}, \tag{234}$$

then the bubble equations for this blob in isolation (*i.e.* with no additional bubbles, black holes or rings) are simply:

$$\mu_i = \sum_{I=1}^{3} k^I, \tag{235}$$

More generally, in any solution satisfying (232), if one finds a blob in which the μ_i are all equal to the same constant, μ_0, then the GH points in the blob must all be arranged in the same way as an isolated black hole but with all the positions scaled by $\mu_0^{-1}\left(\sum_{I=1}^{3} k^I\right)$.

Now consider the full set of N points with $\Delta,\ r_j \ll r_0$. In Sect. 8.2, we solved the last two bubble equations and determined Δ and r_0 in terms of the flux parameters. The remaining bubble equations are then:

$$(N-2-q_i)\,\mu_i + \frac{\alpha_i}{|\vec{r}_N - \vec{r}_i|} - \frac{\beta_i}{|\vec{r}_{(N-1)} - \vec{r}_i|} = \sum_{I=1}^{3}\left((N-2-q_i)\,k^I + \frac{d^I}{2}\right), \tag{236}$$

for $i = 1, \ldots, N-2$. Once again we use the multipole expansion in these equations:

$$(N-2-q_i)\;\mu_i + \frac{(\alpha_i - \beta_i)}{r_0} - \frac{\alpha_i\,\Delta}{r_0^2} = \sum_{I=1}^{3}\left((N-2-q_i)\;k^I + \frac{d^I}{2}\right), \tag{237}$$

It is elementary to show that:

$$\alpha_i - \beta_i = \frac{1}{8}\,(j_L - d^I\,\widehat{Q}_I) + \gamma - \frac{1}{8}\,(N-2-q_i)\,C_{IJK}\,d^I\,k^J\,X^K, \tag{238}$$

where X^I is defined in (222). If one now uses this identity, along with (227) and (230) in (237) one obtains:

$$(N-2-q_i)\ \mu_i - \frac{1}{r_0}\, C_{IJK}\, \left[\frac{1}{8}\ (N-2-q_i)\ d^I\, k^J\, X^K - \left(1-\frac{\alpha_i}{\alpha_0}\right)\ \gamma\right] \approx (N-2-q_i)\ \sum_{I=1}^{3} k^I. \tag{239}$$

Finally, note that:

$$\alpha_0 - \alpha_i = Q\ (N-2-q_i)\ C_{IJK}\ \left[\frac{1}{32}\left(X^I - \frac{1}{Q}d^I\right)\ \left(X^J - \frac{1}{Q}d^J\right)\ k^K + \frac{1}{6}\ k^I\, k^J\, k^K\right], \tag{240}$$

and so the bubble equations (236) reduce to:

$$\begin{aligned}\mu_i &\approx \left(\sum_{I=1}^{3} k^I\right) + \frac{1}{r_0}\, C_{IJK}\ \Bigg[\frac{1}{8}\, d^I\, k^J\, X^K \\ &\qquad -\alpha_0^{-1}\, Q\, \gamma\left(\frac{1}{32}\ \left(X^I - \frac{1}{Q}d^I\right)\ \left(X^J - \frac{1}{Q}d^J\right)\ k^K + \frac{1}{6}\, k^I\, k^J\, k^K\right) \\ &\approx \left(\sum_{I=1}^{3} k^I\right) + \frac{1}{r_0}\, C_{IJK}\ \Bigg[\frac{1}{8}\, d^I\, k^J\, X^K \qquad (241) \\ &\qquad - \alpha_0^{-1}\, Q\, \gamma\left(\frac{1}{32}\, X^I\, X^J\, k^K + \frac{1}{6}\, k^I\, k^J\, k^K\right)\Bigg], \qquad (242)\end{aligned}$$

since we are assuming X^I is large compared to $Q^{-1}d^I$.

Observe that the right-hand side of (241) is independent of i, which means that the first $(N-2)$ GH points satisfy a scaled version of (235) for a isolated, bubbled black hole. Indeed, if $\vec{r}_i^{BH}$ are the positions of a set of GH points satisfying (235) then we can solve (241) by scaling the black hole solution, $\vec{r}_i = \lambda^{-1}\vec{r}_i^{BH}$, where the scale factor is given by:

$$\lambda \approx 1 + \frac{1}{r_0}\ \left(\sum_{I=1}^{3} k^I\right)^{-1}\ C_{IJK}\ \left[\frac{1}{8}\, d^I\, k^J\, X^K - \alpha_0^{-1}\, Q\, \gamma\left(\frac{1}{32}\, X^I\, X^J\, k^K + \frac{1}{6}\, k^I\, k^J\, k^K\right)\right]. \tag{243}$$

Notice that as one approaches the critical "merger" value, at which $\Omega = j_L - d^I\widehat{Q}_I = 0$, (243) implies that the distance, r_0, must also scale as λ^{-1}. Therefore the merger process will typically involve sending $r_0 \to 0$ while respecting the assumptions made in our approximations $(\Delta,\ r_i \ll r_0)$. The result will be a "scaling solution" in which all distances in the GH base are vanishing while preserving their relative sizes.

In [53], this picture of the generic merger process was verified by making quite a number of numerical computations[24]; we urge the curious reader to refer to that paper for more details. In Sect. 8.4, we will only present one very simple scaling solution, which illustrates the physics of these mergers.

An important exception to the foregoing analysis arises when the term proportional to r_0^{-1} in (241) vanishes to leading order. In particular, this happens if we violate one of the assumptions of our analysis, namely, if one has:

$$X^I \equiv 2\, f^I - d^I - 4\,(N-2)\, k^I \approx 0, \tag{244}$$

to leading order order in $Q^{-1}d^I$. If X^I vanishes one can see that, to leading order, the merger condition is satisfied:

$$\begin{aligned} \Omega &\equiv j_L - d^I\, \widehat{Q}_I \\ &= \frac{1}{8}\, C_{IJK}\, d^I\, \left[X^J X^K - \frac{1}{3}\, Q^{-2}\,(4\,Q-1)\, d^J d^K - 16\, k^J k^K \right] \\ &\approx 0, \end{aligned} \tag{245}$$

and so one must have $r_0 \to 0$. However, the foregoing analysis is no longer valid, and so the merger will not necessarily result in a scaling solution.

An important example of this occurs when k^I, d^I, and f^I are all parallel:

$$k^I = k\, u^I, \qquad d^I = d\, u^I, \qquad f^I = f\, u^I, \tag{246}$$

for some fixed u^I. Then the merger condition (245) is satisfied to leading order, only when $X \equiv (2\,f - d - 4\,(N-2)k)$ vanishes.

For non-parallel fluxes it is possible to satisfy the merger condition, (245), while keeping X^I large, and the result is a scaling solution.

Even if it looks like irreversible mergers progress until the final size on the base vanishes, this is an artifact of working in a classical limit and ignoring the quantization of the fluxes. After taking this into account, we can see from (241) that r_0 cannot be taken continuously to zero because the d^I, f^I, X^I, and k^I are integers of half-integers. Hence, the final result of an irreversible merger is a microstate of a high, but finite, redshift and whose throat only becomes infinite in the classical limit.

In order to find the maximum depth of the throat, one has to find the smallest allowed value for the size of the ensemble of GH points in the $\mathbb{R}^3$ base of the GH space. During the irreversible merger all the distances scale, the size of the ensemble of points will be approximately equal to the distance between the ring blob and the black hole blob, which is given by (230). Since $j_L - d^I \widehat{Q}_I$ is quantized, the minimal size of the ensemble of GH points is given by:

[24] A merger was tracked through a range where the scale factor, λ, varied from about 4 to well over 600. It was also verified that this scaling behavior is not an artifact of axial symmetry. Moreover, in several numerical simulations the GH points of the black-hole blob were arranged along a symmetry axis but the bubbled ring approached the black-hole blob at various angles to this axis; the scaling behavior was essentially unmodified by varying the angle of approach.

$$r|_{\min} \approx \frac{1}{d^1+d^2+d^3}. \quad (247)$$

More generally, in the scaling limit, the GH size of a solution with left-moving angular momentum J_L is

$$r|_{\min} \approx \frac{J_L}{d^1+d^2+d^3}. \quad (248)$$

Since the d^I scale like the square-roots of the ring charges, we can see that in the classical limit, $r|_{\min}$ becomes zero and the throat becomes infinite.

8.4 Numerical Results for a Simple Merger

Given that most of the numerical investigations, and most of the derivations we have discussed above use black hole microstate made from a very large number of points, it is quite hard to illustrate explicitly the details of a microstate merger.

To do this, it is much more pedagogical to investigate a black hole microstate that is made from three points, of GH charges $-n$, $2n+1$, and $-n$, and its merger with the black ring microstate of GH charges $-Q$ and $+Q$. This black-hole microstate can be obtained by redistributing the position of the GH points inside the black-hole blob considered in Sect. 8.3, putting all the $+1$ charges together and putting half of the -1 charges together on one side of the positive center and the other half on the other side[25]

We consider a configuration with 5 GH centers of charges

$$q_1 = -12,\ q_2 = 25,\ q_3 = -12,\ q_4 = -20,\ q_5 = 20. \quad (249)$$

The first three points give the black-hole "blob," which can be thought as coming from a blob of $N-2=49$ points upon redistributing the GH points as described above; the k^I parameters of the black hole points are

$$k_1^I = q_1 \hat{k}_0^I,\ k_2^I = q_2 \hat{k}_0^I,\ k_3^I = q_3 \hat{k}_0^I, \quad (250)$$

where $\hat{k}_0^I$ is the average of the k^I over the black-hole points, defined in (193). To merge the ring and the black hole microstates, we have varied $\hat{k}_0^2$ keeping $\hat{k}_0^1$ and $\hat{k}_0^3$ fixed:

$$\hat{k}_0^1 = \frac{5}{2},\ \hat{k}_0^3 = \frac{1}{3}, \quad (251)$$

We have also kept fixed the ring parameters f^I and d^I:

$$d^1 = 100,\ d^2 = 130,\ d^3 = 80,\ f^1 = f^2 = 160,\ f^3 = 350 \quad (252)$$

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[25] Since the k parameters on the black-hole points are the same, the bubble equations give no obstruction to moving black-hole centers of the same GH charge on top of each other.

The relation between these parameters and the k^I of the ring is given in (195), where $N-2$ (the sum of $|q_i|$ for the black hole points) is now $|q_1|+|q_2|+|q_3|=49$.

The charges and J_R angular momentum of the solutions are approximately

$$Q_1 \approx 68.4\times 10^3,\ Q_2 \approx 55.8\times 10^3,\ Q_3 \approx 112.8\times 10^3,\ J_R \approx 3.53\times 10^7, \quad (253)$$

while J_L goes to zero as the solution becomes deeper and deeper.

Solving the bubble equations (142) numerically, one obtains the positions x_i of the five points as a function of $\hat{k}_0^2$. As we can see from the Table 2, a very small increase in the value of $\hat{k}_0^2$ causes a huge change in the positions of the points on the base. If we were merging real black holes and real black rings, this increase would correspond to the black hole and the black ring merging. For the microstates, this results in the scaling described above: all the distances on the base become smaller, but their ratios remain fixed.

Table 2 Distances between points in the scaling regime

	$\hat{k}_0^2$	x_4-x_3	$\frac{x_4-x_3}{x_2-x_1}$	$\frac{x_2-x_1}{x_3-x_2}$	$\frac{x_2-x_1}{x_5-x_4}$	J_L	$\mathscr{H}$
0	3.0833	175.5	2225	1.001	2.987	215983	0.275
1	3.1667	23.8	2069	1.001	3.215	29316	0.278
2	3.175	8.65	2054	1.001	3.239	10650	0.279
3	3.1775	4.10	2049	1.001	3.246	5050	0.279
4	3.178	3.19	2048	1.001	3.248	3930	0.279
5	3.17833	2.59	2048	1.001	3.249	3183	0.279
6	3.17867	1.98	2047	1.001	3.250	2437	0.279
7	3.1795	0.463	2046	1.001	3.252	570	0.279
8	3.17967	0.160	2045	1.001	3.253	197	0.279

The parameter $\mathscr{H} \equiv \frac{Q_1Q_2Q_3-J_R^2/4}{Q_1Q_2Q_3}$ measures how far away the angular momentum of the resulting solution is from the angular momentum of the maximally spinning black hole with identical charges. The value of $\hat{k}_0^2$ is varied to produce the merger, and the other parameters of the configuration are kept fixed: $Q=20,\ q_1=q_3=-12,\ q_2=25,\ \hat{k}_0^1=\frac{5}{2},\ \hat{k}_0^3=\frac{1}{3},\ d^1=100,\ d^2=130,\ d^3=80,\ f^1=f^2=160,\ f^3=350$. Both the charges and J_R remain approximately constant, with $J_R\approx 3.53\times 10^7$.

Checking analytically that these solutions have no closed time-like curves is not that straightforward, since the quantities in (140) have several hundred terms. However, in [53] it was found numerically that such closed time-like curves are absent and that the equations (140) are satisfied throughout the scaling solution.

8.5 The Metric Structure of the Deep Microstates

The physical metric is given by (28) and (29) and the physical distances are related to the coordinate distances on the the $\mathbb{R}^3$ base of the GH space, $d\vec{y}\cdot d\vec{y}$ via:

$$ds^2 = (Z_1 Z_2 Z_3)^{1/3}\, V\, d\vec{y}\cdot d\vec{y}. \tag{254}$$

The physical lengths are thus determined by the functions, $Z_I V$, and if one has:

$$(Z_1 Z_2 Z_3)^{1/3}\, V \sim \frac{1}{r^2}, \tag{255}$$

then the solution looks as an $AdS_2 \times S^3$ black hole throat. In the region where the constants in the harmonic functions become important, this throat turns into an asymptotically flat $\mathbb{R}^{(4,1)}$ region. Near the GH centers that give the black-hole bubbles, the function $Z_1 Z_2 Z_3$ becomes constant. This corresponds to the black-hole throat "capping off". As the GH points get closer in the base, the region where (255) is valid becomes larger, and hence the throat becomes longer.

As one may intuitively expect, in a scaling solution the ring is always in the throat of the black hole. Indeed, the term "1" on the right-hand side of (243) originates from the constant terms in L_I and M, defined in (129). In the scaling regime, this term is sub-leading, which implies the ring is in a region where the 1 in the L_I (and hence the Z_I) is also sub-leading. Hence, the ring lies in the AdS throat of the black-hole blob.

Increasing the scale factor, λ, in (243) means that the bubbles localize in a smaller and smaller region of the GH base, which means that the throat is getting longer and longer. The physical circumference of the throat is fixed by the charges and the angular momentum, and remains finite even though the blob is shrinking on the GH base. Throughout the scaling, the throat becomes deeper and deeper; the ring remains in the throat, and also descends deeper and deeper into it, in direct proportion to the overall depth of the throat.

On a more mechanistic level, the physical distance through the blob and the physical distance from the blob to the ring are controlled by integrals of the form:

$$\int (Z_1 Z_2 Z_3\, V^3)^{1/6}\, d\ell. \tag{256}$$

In the throat, the behavior of this function is given by (255), and this integral is logarithmically divergent as $r \to 0$. However, the Z_I limit to finite values at $\vec{r} = \vec{r}_j$ and between two very close neighboring GH points in the blob, the integral has a dominant contribution of the form

$$C_0 \int |(x-x_i)(x-x_j)|^{-1/2}\, dx, \tag{257}$$

for some constant, C_0, determined by the flux parameters. This integral is finite and indeed is equal to $C_0\, \pi$. Thus we see that the throat gets very long but then caps off with bubbles of finite physical size.

8.6 Are Deep Microstates Dual to Typical Boundary Microstates?

As we have seen in Sect. 8.5, the throats of the deep microstates become infinite in the classical limit. Nevertheless, taking into account flux quantization one can find that the GH radius of microstates does not go all the way to zero but to a finite value (247), which corresponds to setting $J_L = 1$.

One can estimate the energy gap of the solution by considering the lightest possible state at the bottom of the throat and estimating its energy as seen from infinity. The lightest massive particle one can put on the bottom of the throat is not a Planck-mass object, but a Kaluza-Klein mode on the S^3. Its mass is

$$m_{KK} = \frac{1}{R_{S^3}} = \frac{1}{(Q_1 Q_2 Q_3)^{\frac{1}{6}}} \qquad (258)$$

and therefore the mass gap in a microstate of size $r_{\min}$ in the GH base is:

$$\Delta E_{r_0} = m_{KK}\sqrt{g_{00}}|_{r=r_{\min}} = m_{KK}(Z_1 Z_2 Z_3)^{-1/3}|_{r=r_{\min}} = \frac{r_{\min}}{(Q_1 Q_5 Q_P)^{1/2}}. \qquad (259)$$

For a ring-hole merger, $r_{\min}$ depends on the sum of the d^I, and so its relation with the total charges of the system is not straightforward. Nevertheless, we can consider a regime where $Q_1 \sim Q_5 > Q_P$, and in this regime the dipole charge that dominates the sum in (248) is $d^3 \approx \sqrt{\frac{Q_1 Q_5}{Q_P}}$. Hence

$$r_{\min} = \frac{J_L}{d^3} \approx J_L \sqrt{\frac{Q_P}{Q_1 Q_5}}, \qquad (260)$$

Exercise 17. *Show that the mass gap for a KK mode sitting on the bottom of the throat at $r \sim r_{\min}$ is*

$$\Delta E_{r_{\min}} \approx \frac{J_L}{Q_1 Q_5}. \qquad (261)$$

This M-theory frame calculation is done in the limit $Q_1 \sim Q_5 > Q_P$, which is the limit in which the solution, when put into the D1-D5-P duality frame, becomes asymptotically $AdS_3 \times S^3 \times T^4$. As shown in [56], in this limit $d^1 + d^2 + d^3 \approx d^3$, which justifies going from (248) to (260).

For $J_L = 1$, the mass gap computed in the bulk (261) matches the charge dependence of the mass gap of the black hole [120]. Moreover, this mass gap should also match the mass gap of the dual microstate in the D1-D5 CFT.

As is well known (see [39, 40] for reviews), the states of this CFT can be characterized by various ways of breaking a long effective string of length $N_1 N_5$ into component strings. BPS momentum modes on these component strings carry J_R. The fermion zero modes of each component string allow it in addition to carry one unit of J_L. The typical CFT microstates that contribute to the entropy of the three-charge black hole have one component string [73]; microstates dual to objects that have a macroscopically large J_L have the effective string broken into many component strings [9, 11, 56].

Hence, the only way a system can have a large J_L is to be have many component strings. The CFT mass gap corresponds to exciting the longest component string and is proportional to the inverse of its length.

The formula (261) immediately suggests what the dual of a deep microstate should be. Consider a long effective string of length N_1N_5 broken into J_L component strings of equal length. Each component string can carry one unit of left-moving angular momentum, totaling up to J_L. The length of each component string is

$$l_{\text{component}} = \frac{N_1N_5}{J_L}, \tag{262}$$

and hence the CFT mass gap is

$$\Delta E_{CFT} \approx \frac{J_L}{N_1N_5}. \tag{263}$$

This agrees with *both* the J_L dependence and the dependence on the charges of the gap computed in the bulk. While we have been cavalier about various numerical factors of order one, the agreement that we have found suggests that deep microstates of angular momentum J_L are dual to CFT states with J_L component strings. If this is true, then the deepest microstates, which have $J_L = 1$, correspond to states that have only one component string, of length N_1N_5. This is a feature that typical microstates of the three-charge black hole have, and the fact that deep microstates share this feature is quite remarkable.

Our analysis here has been rather heuristic. It would be very interesting to examine this issue in greater depth by finding, at least, approximate solutions to the wave equation in these backgrounds and performing an analysis along the lines of [9, 11].

9 Implications for Black-Hole Physics

9.1 Microstate Geometries

As we have seen, string theory contains a huge number of smooth configurations that have the same charges and asymptotics as the three-charge BPS black hole in five dimensions. Counting these configurations, or relating them to the states of the boundary CFT, will allow one to prove or disprove the claim that black holes in string theory are not fundamental objects, but rather a statistical way to describe an ensemble of black-hole-sized configurations with no horizon and with unitary scattering. This will help in establishing the answer to the key question "What is the AdS-CFT dual of the states of the D1-D5-P system?" Nevertheless, even if a definitive answer may be hard to establish and prove, it is well worth exploring in more detail the three (or four) possible answers to this question, particularly in light of our current understanding of black hole microstates:

Possibility 1: One Bulk Solution Dual to Many Boundary Microstates

It is possible that some of the states of the CFT, and in particular the typical ones (whose counting gives the black hole entropy) do not have individual bulk duals, while some other states do. However this runs counter to all our experience with the *AdS*-CFT correspondence: In all the examples that have been extensively studied and well-understood (like the D1-D5 system, Polchinski-Strassler [67], giant gravitons and LLM [117, 121, 122, 123, 124], the D4-NS5 system [125, 126]) the *AdS*-CFT correspondence relates boundary states to bulk states and boundary vacua to bulk vacua.

It is logically possible that, for the D1-D5-P system only, the path integrals in the bulk and on the boundary are related in the standard way via the *AdS*-CFT correspondence, and yet all the boundary states that give the CFT entropy are mapped into one black hole solution in the bulk. This possibility is depicted in Fig. 8. However, this possibility raises a lot of questions. First, why would the D1-D5-P system be different from all the other systems mentioned above. Moreover, from the microscopic (or CFT) perspective, there is nothing special about having three charges: One can map the boundary states in the D1-D5-KKM system in *four* dimensions to the corresponding bulk microstates [46, 127, 128]. The only reason for which the D1-D5-P system would be different from all the other systems would be the fact that it has the right amount of charges to create a macroscopically large event horizon in five dimensions. To have such divergently different behavior for the D1-D5-P system in five dimensions would be, depending on one's taste, either very deep or, more probably, very bizarre.

Even if the typical states of the three-charge system correspond to one single black hole, we have seen that besides this black hole there exists a huge number of smooth solutions that also are dual to individual states of this CFT. Hence, according to Possibility 1, some states of the CFT would have individual bulk duals and some

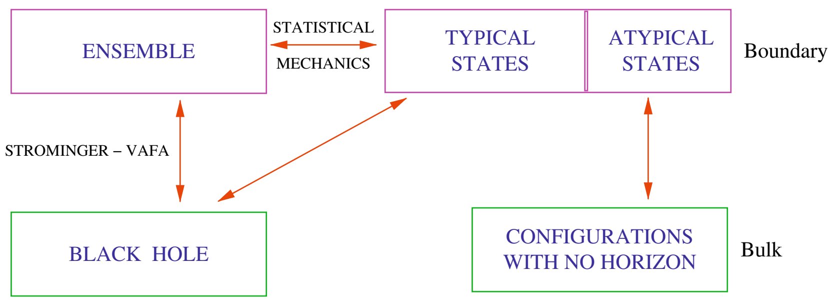

Fig. 8 A schematic description of Possibility 1

others would not (they would be dual as an ensemble to the black hole). This distinction is very unnatural. One might explain this if the states dual to the black hole and the ones dual to microstate geometries are in different sectors of the CFT, but this is simply not the case. We have seen in Sect. 8.6 that the deep bulk microstates correspond to boundary states that have one (or several) long component string(s). Hence, they belong to the same CFT sector as the typical microstates. If typical microstates did not have individual bulk duals, then in the same sector of the CFT, we would have both states with a bulk dual and states without one. While not obviously wrong, this appears, at least, dubious and unjustifiable from the point of view of the CFT.

Possibility 2: Typical Bulk Microstate Very Similar to Black Hole

It is possible that all the states of the CFT are dual to geometries in the bulk, but the typical states are dual to geometries that have a horizon, and that only differ from the classical black hole by some Planck-sized fuzz near the singularity. This situation is depicted in Fig. 9.

This also has a few problems. First, there are arguments, [27, 28], that if the microstates of the black hole only differ from the classical geometry near the singularity, it does not solve the information paradox. Putting such arguments on one side, there is a more obvious objection: Possibility 2 means that typical microstates would have horizons, and so it would seem that one would have to ascribe an entropy to each microstate, which violates one of the principles of statistical mechanics. A counterargument here is to observe that one can always ascribe an ad hoc entropy to a microstate of any system simply by counting the number of states with the same macroscopic properties. What really distinguishes a microstate from an ensemble is that one has complete knowledge of the state of the former and that one has lost some knowledge of the state in the latter. The counterargument asserts that the presence of the horizon does not necessarily indicate information loss and that the complete information might ultimately be extracted from something like the Hawking radiation. Thus microstates could have a horizon if information is somehow stored and not lost in the black hole. This is a tenable viewpoint and it is favored by a number

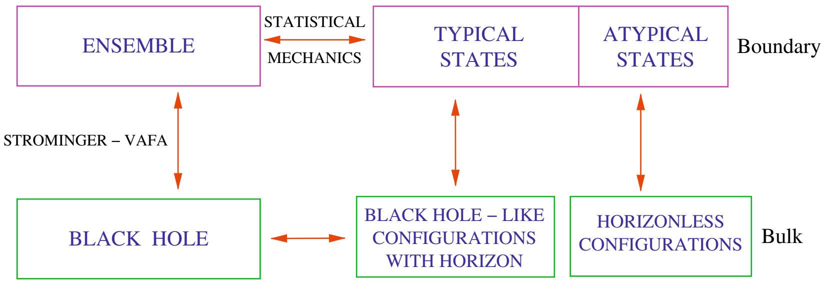

Fig. 9 A schematic description of Possibility 2

of relativists, but it defers the issue of how one decodes the microstate information to some unknown future physics, whereas string theory appears to be pointing to a very interesting answer in the present.

There is also one of the objections raised in possibility 1: We have seen that some CFT states corresponding to long component strings are dual to deep microstates that have no horizon. If the second possibility is correct, then other states in *the same sector* of the CFT would be dual to geometries that have a horizon and a singularity and are therefore drastically different. Moreover, for extremal black holes, the distance to the horizon is infinite, while the distance to the cap of the microstates is finite (though divergent in the classical limit). Hence, in the same sector of the CFT, some states would be dual to supergravity solutions with an infinite throat, while others would be dual to solutions with a finite throat. This again appears quite dubious from the point of view of the CFT.

One can also think about obtaining the bulk microstate geometries by starting from a weak-coupling microstate (which is a certain configuration of strings and branes) and increasing the string coupling. During this process, we can imagine measuring the distance to the configuration. If a horizon forms, then this distance would jump from being finite to being infinite. However, for the smooth microstates, this distance is always a continuous function of the string coupling and never becomes infinite. While the infinite jump of the length of the throat is a puzzling phenomenon, equally puzzling is the fact that only some microstates would have this feature, while some very similar ones would not.[26]

Possibility 3: Typical Bulk Microstate Differs from Black Hole at the Scale of the Horizon

It is possible that all boundary microstates are dual to horizonless configurations. The classical black hole geometry is only a thermodynamic description of the physics, which stops being valid at the scale of the horizon, much like fluid mechanics stops being a good description of a gas at scales of order the mean free path. For physics at the horizon scale, one cannot rely on the thermodynamic description and has to use a "statistical" description in terms of a large number of microstates. This possibility is depicted in Fig. 10.

Since these microstates have no horizon, they have unitary scattering but it takes a test particle a very long time to escape from this microstate. Hence, if this possibility is correct, the information paradox is reduced to nothing but an artifact of using a thermodynamic description beyond its regime of validity. This possibility now splits into two options, having to do with the appropriate description of the typical black hole microstates:

Possibility 3A: Typical microstates cannot be described in supergravity and require the full force of string theory.
Possibility 3B: Typical microstates can be described in supergravity.

[26] We thank Samir Mathur for pointing out this argument to us.

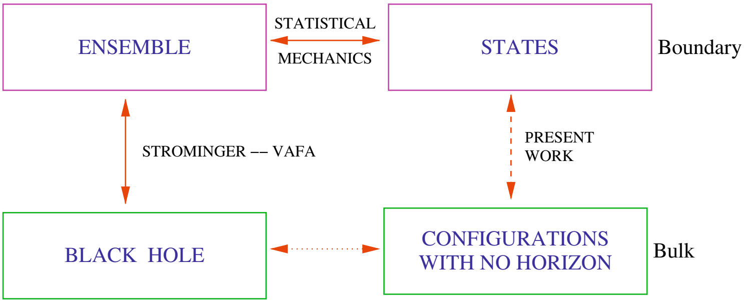

Fig. 10 A schematic description of Possibility 3

As we cannot, yet, explore or count large strongly-interacting horizon-sized configurations of branes and strings using our current string theory technology, Possibility 3A would be more challenging to establish or analyze. We therefore need to examine Possibility 3B in great detail to see if it is true, or at least determine the extent to which supergravity can be used. One way to do this is by counting the microstates, using for example counting techniques of the type used in [29, 30, 31, 32]. Another approach is to find the exact (or even approximate) dictionary between the states of the CFT and the bubbled geometries in the bulk. Anticipating (or perhaps speculating) a bit, one could imagine that, as a result of this investigation, one could relate the number of bubbles of a deep microstate to the distribution of the momentum on the long component string of the dual CFT state. Such a relation (which could in principle be obtained using scattering experiments as in [11, 42]) would indicate whether typical bulk microstates have large bubbles or Planck-sized bubbles and would help distinguish between Possibilities 3A and 3B.

One of the interesting questions that needs to be addressed here is: *What about non-extremal black holes?* All the arguments presented in this review in favor of the third possibility have been based on supersymmetric black holes, and one can legitimately argue that even if these black holes describe an ensemble of smooth horizonless configurations, it may be that non-supersymmetric black holes (like the ones we have in the real world) are fundamental objects and not ensembles. The arguments put forth to support Possibility 3 for non-extremal black holes are rather more limited. Indeed, on a technical level, it is much more difficult to find non-supersymmetric, smooth microstate geometries, but some progress has been made [103]. There are nevertheless some interesting physical arguments based primarily on the phenomenon of charge fractionation.

The idea of charge fractionation [129, 130] is most simply illustrated by the fact that when you put N_1 D1 branes (or strings) in a periodic box of length L, then the lowest mass excitation carried by this system is not of order L^{-1} but of order $(N_1L)^{-1}$. The explanation is that the branes develop multi-wound states with the longest effective length being of order N_1L. Similarly, but via a rather more complex mechanism, the lowest mass excitations of the D1-D5 system vary as $(N_1N_5)^{-1}$. This is called charge fractionation. It is this phenomenon that leads to the CFT mass gap given in (263). The other important consequence of fractionation is that the corresponding "largest" natural physical length scale of the system grows as N_1N_5. One of the crucial physical questions is how does the "typical" length scale grow with charge. That is, what is the physical scale of the most likely (or typical) configuration. It is believed that this will grow as some positive power of the underlying charges, and this is the fundamental reason why it is expected that microstate geometries are "large" compared to the Planck scale and that microstate geometries are not just relevant within a few Planck units of the singularity but extend to the location of the classical horizon.

This argument can be extended to non-BPS systems. Configurations of multiple species of branes also exhibit fractionation. For this reason, it is believed that, given a certain energy budget, the way to get most entropy is to make brane-antibrane pairs of different sorts.[27] Putting together these different kinds of branes creates a system with very light (fractionated) modes, whose mass is much much lower than the Planck scale. These modes can then "extend" all the way to the horizon and have to be taken into account when discussing physics at this scale.

One of the counterarguments to the third possibility is that one can collapse a shell of dust and create a horizon at very weak curvatures, long before the black hole singularity forms. Moreover, the larger the mass, the longer will be the time elapsed between the formation of the horizon and the singularity. Hence, it naively appears that the horizon cannot possibly be destroyed by effects coming from a singularity that is so far away. Nevertheless, if fractionation gives the correct physics, then one can argue that as the mass of the incoming shell increases, the number of brane-antibrane pairs that are created becomes larger, and hence the mass of the "fractionated" modes becomes smaller; these modes will then affect the physics at larger and larger scales, which can be argued to be of order the horizon size. In this picture, the collapsing shell would reach a region where a whole new set of very light degrees of freedom exist. Since these "fractionated" degrees of freedom have a much larger entropy, the shell will dump all its energy into these modes, which would then expand to the horizon and destroy the classical geometry up to this scale. More details in support of these arguments can be found in [27, 28].

On the other hand, one may hope to preserve the status quo for non-extremal black holes by arguing that fractionation is a phenomenon that is based on weakly coupled D-brane physics and is not necessarily valid in the range of parameters where the black hole exists. This, however, leaves one with the problem of explaining why fractionation appears to be occurring in extremal black holes and why non-BPS black holes should be any different. Indeed, if the classical solution for the extremal black hole is proven to give an incorrect description of the physics at the horizon when embedded into a quantum theory of gravity, it is hard to believe that other similar, non-extremal solutions will give a correct description of the physics

[27] This idea has been used in formulating microscopic brane-antibrane models for near-BPS black holes [131, 132] and for black branes [133], and has recently received a beautiful confirmation in the microscopic calculation of the entropy of extremal non-BPS black holes [134]. It has also been applied to cosmology [135] and to understanding the Gregory Laflamme instability [136, 137] microscopically [138, 139].

at the horizon. It will be much more reasonable to accept that all the classical black hole solutions are thermodynamic descriptions of the physics, which break down at the scale of the horizon.

The most direct support for the smooth microstate structure of non-extremal black holes would be the construction and counting of smooth, non-extremal geometries generalizing those presented here, like those constructed in [103]. Such constructions are notoriously difficult and, barring a technical miracle in the construction of non-BPS solutions, it is hard to hope that there will be a complete classification of such geometries in the near future. On the other hand, it is instructive and encouraging to recall the developments that happened shortly after the original state counting arguments of Strominger and Vafa for BPS black holes: There was a lot of analysis of near-BPS configurations and confirmation that the results could be generalized perturbatively to near-BPS states with small numbers of anti-branes. This might prove fruitful here and would certainly be very useful in showing that generic smooth microstate geometries are not special properties of BPS objects. It would thus be interesting to try, either perturbatively, or perhaps through microstate mergers, to create near-BPS geometries.

Finally, the fact that the classical black hole solution does not describe the physics at the scale of the horizon seems to contradict the expectation that this solution should be valid there since its curvature is very small. There are, however, circumstances in which this expectation can prove wrong. First, if a solution has a singularity, it oftentimes does not give the correct physics even at a very large distance away from this singularity because the boundary conditions at the singularity generate incorrect physics even in regions where the curvature is very low. Such solutions therefore have to be discarded. A few examples of such solutions are the Polchinski-Strassler flow [67] without brane polarization [140, 150], or the singular KK giant graviton [121, 122, 123, 124, 141]. The reason why we do not automatically discard black hole solutions is that their singularities are hidden behind horizons and sensible boundary conditions can be imposed at the horizon. However, this does not imply that all solutions with singularities behind horizons must be good: It only shows that they should not be discarded a priori, without further investigation. What we have tried to show is that if the third possibility is correct then the investigation indicates that the classical BPS black-hole solution should not be trusted to give a good description of the physics at the scale of the horizon.

9.2 A Simple Analogy

To understand Possibility 3 a little better, it is instructive to recall the physics of a gas, and to propose an analogy between the various descriptions of a black hole and the various descriptions of this gas.

For scales larger than the mean free path, a gas can be described by thermodynamics, or by fluid mechanics. At scales below the mean free path, the thermodynamic description breaks down, and one has to use a classical statistical description, in which one assumes all the molecules behave like small colliding balls. When the molecules are very close to each other, this classical statistical description breaks down, and we have to describe the states of this gas quantum mechanically. Moreover, when the temperature becomes too high, the internal degrees of freedom of the molecules become excited, and they cannot be treated as small balls. There are many features, such as shot noise or Brownian motion, that are not seen by the thermodynamic description, but can be read off from the classical statistical description. There are also features that can only be seen in the full quantum statistical description, such as Bose-Einstein condensation.

For black holes, if Possibilities 3A or 3B are correct, then the *AdS*-CFT correspondence relates quantum states to quantum states, and we expect the bulk dual of a given boundary state to be some complicated quantum superposition of horizonless configurations. Unfortunately, studying complicated superpositions of geometries is almost impossible, so one might be tempted to conclude that even if Possibilities 3A or 3B are correct, there is probably no new physics one can learn from it, except for an abstract paradigm for a solution to the information paradox. Nevertheless, we can argue by analogy to a gas of particles that this is not the case.

Consider a basis for the Hilbert space of the bulk configurations. If this basis is made of coherent states, some of the states in this basis will have a semiclassical description in terms of a supergravity background. This would be very similar to the situation explored in [117], where bubbled geometries correspond to coherent CFT states. The supergravity solutions we have discussed in these notes are examples of such coherent states. The main difference between the Possibility 3A and 3B has to do with whether the coherent states that form a basis of the Hilbert space can be described using supergravity or whether one has to use string theory to describe them. By analogy with the gas, this is the difference between the regime where the simple "colliding ball" model is valid, and the regime where one excites internal degrees of freedom of the molecules.

If supergravity is a good description of most of the coherent states, we can argue that we have constructed the black hole analogue of the classical statistical description of an ideal gas. Even if most of the coherent states can only be described in a full string-theoretic framework, one can still hope that this will give the analogue of a more complicated, classical statistical description of the gas. Both these descriptions are more complete than the thermodynamic description, and for the gas, they capture physics that the thermodynamic description overlooks. Apart from solving the information problem, it would be very interesting to identify precisely what this physics is for a black hole. Indeed, as we will explain below, it might lead to some testable signature of string theory.

On the other hand, the black-hole analogue of the quantum statistical description involves a complicated and hard-to-study quantum superposition of microstates, and is therefore outside our present theoretical grasp. One can speculate, again in analogy with the ideal gas, that there are probably interesting physical phenomena that can only be captured by this description and not by the classical statistical description.

We should also note that in [23] it has been argued that from the point of view of the dual CFT, the difference between the typical microstates and the classical black hole solution can only be discerned by doing a very atypical measurement or waiting for a very long time.[28] This is analogous to the case of a gas, where if one waits for a very long time, of order the Poincaré recurrence time, one will observe spikes in the pressure coming from very unlikely events, such as a very large number of molecules hitting the wall at the same time. In the thermodynamic approximation one ignores the small energy gap between microstates, and such phenomena are not visible. The fact that the classical black hole geometry has an infinite throat and no mass gap implies that this geometry will not display such fluctuations at very large time-scales. Since the CFT does have a mass gap, and fluctuations at large scales occur, one can argue [144] that the black hole gives a thermodynamic description of the physics and not a microscopic one.

Since, by standard AdS-CFT arguments, a long time on the boundary corresponds to a large distance into the bulk, one can argue that atypical CFT measurements involving very long times correspond in the bulk to propagators that reach very close to the black hole horizon [23]. Hence, this supports the intuition that one can distinguish between different microstates by making experiments at the scale of the horizon. Moreover, in a gas one can distinguish between the ensemble and the microstates by making experiments at scales smaller than the mean free path. At this scale, the thermodynamic description breaks down, and new phenomena that cannot be captured by thermodynamics appear. By analogy, for the black hole we have argued that the scale where thermodynamics breaks down is that of the horizon. Therefore, both our arguments and the arguments of [23] indicate that experiments made *at the scale of the horizon* should distinguish between a microstate and the classical solution. While from the point of the dual CFT these experiments appear to be very atypical, they might not be so atypical from the point of view of the dual bulk. It would certainly be very interesting to propose and analyze in more detail such gedanken experiments and explore more thoroughly the implications of this fact.

The whole problem with finding experimental or observational tests of string theory is that the string scale and the Planck scale are so far out of reach of present accelerations. However, the ideas of fractionation and the present ideas about the microstate structure of black holes show us that we can get stringy effects on very large length scales. It would obviously be very exciting if we could make black holes at the LHC and thereby test these ideas, but even if this were not to happen, we may still be able to see some signature of stringy black holes within the next decade. Indeed, the gravitational wave detectors LIGO and LISA are very likely to detect the gravitational "ring-down" of merging black holes within the next few years and, while the underlying computations will be extremely difficult, one might reasonably hope that the microstate structure arising from string theory could lead to a new, detectable and recognizable signature in the LIGO or LISA data.

[28] See [142, 143] for other interesting work in this direction.

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Black Hole Entropy and Quantum Information

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Abstract We review some recently established connections between the mathematics of black hole entropy in string theory and that of multipartite entanglement in quantum information theory. In the case of $N = 2$ black holes and the entanglement of three qubits, the quartic $[SL(2)]^3$ invariant, Cayley's hyperdeterminant, provides both the black hole entropy and the measure of tripartite entanglement. In the case of $N = 8$ black holes and the entanglement of seven qubits, the quartic E_7 invariant of Cartan provides both the black hole entropy and the measure of a particular tripartite entanglement encoded in the Fano plane.

1 Black Holes and Qubits

It sometimes happens that two very different areas of theoretical physics share the same mathematics. This may eventually lead to the realization that they are, in fact, dual descriptions of the same physical phenomena, or it may not. Either way, it frequently leads to new insights in both areas. In this paper, the two areas in question are black hole entropy in string theory and qubit entanglement in quantum information theory. Going one way, we shall learn that the entropy of the so-called STU $N = 2$ black hole is given by the "hyperdeterminant", a quantity first introduced by Cayley in 1845 and which describes the tripartite entanglement of three qubits [1, 2, 3]. Going the other way, we discover that the exceptional group E_7, the U-duality group of $N = 8$ supergravity, plays a part in the tripartite entanglement of seven qubits [4, 5].

We begin in Sect. 2 with an interesting subsector of string compactification to four dimensions which is provided by the *STU* model whose low energy limit is

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described by $N = 2$ supergravity coupled to three vector multiplets. One may regard it as a truncation of an $N = 4$ theory obtained by compactifying the heterotic string on T^6 where S,T,U correspond to the dilaton/axion, complex Kahler form and complex structure fields, respectively. It exhibits an $SL(2,Z)_S$ strong/weak coupling duality and an $SL(2,Z)_T \times SL(2,Z)_U$ target space duality. By string/string duality, this is equivalent to a Type IIA string on $K3 \times T^2$ with S and T exchanging roles [6, 7, 8]. Moreover, by mirror symmetry this is in turn equivalent to a Type IIB string on the mirror manifold with T and U exchanging roles. Another way to obtain this model is by truncation of the N $= 8$ theory that results from T^7 compactification of M-theory. Either way, the truncated theory has a combined $[SL(2,Z)]^3$ duality and complete $S-T-U$ triality symmetry [9]. Alternatively, one may simply start with this $N = 2$ theory directly as an interesting four-dimensional supergravity in its own right, as described in Sect. 2.

The model admits extremal black holes solutions carrying four electric and magnetic charges, and we organize these 8 charges into the $2 \times 2 \times 2$ *hypermatrix*, a_{ABD}, and display the $S-T-U$ symmetric Bogomolnyi mass formula [9]. Associated with this hypermatrix is a *hyperdeterminant*, Det a_{ABD}, discussed in Sect. 3, first introduced by Cayley in 1845 [10]. The black hole entropy, first calculated in [11], is quartic in the charges and must be invariant under $[SL(2,Z)]^3$ and under triality. The main result of Sect. 4 is to show [1] that this entropy is given by the square root of Cayley's hyperdeterminant:

$$S = \pi\sqrt{|\mathrm{Det}\, a_{ABD}|}. \tag{1}$$

The hyperdeterminant also makes it appearance in quantum information theory [12]. Let the three qubit system ABD (Alice, Bob amd Daisy) be in a pure state $|\Psi\rangle$, and let the components of $|\Psi\rangle$ in the standard basis be a_{ABD}:

$$|\Psi\rangle = a_{ABD}|ABD\rangle \tag{2}$$

or

$$\begin{aligned}|\Psi\rangle &= a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle \\ &\quad + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle\end{aligned} \tag{3}$$

Then the three-way entanglement of the three qubits A, B and D is given by the *3-tangle* [13]

$$\tau_3(ABD) = 4|\mathrm{Det}\, a_{ABD}|. \tag{4}$$

The 3-tangle is maximal for the GHZ state $|000\rangle + |111\rangle$ [14] and vanishes for the states $p|100\rangle + q|010\rangle + r|001\rangle$. The relation between three qubit quantum entanglement and the Cayley hyperdeterminant was pointed out by Miyake and Wadati [12].

As far as we can tell [1], the appearance of the Cayley hyperdeterminant in these two different contexts of stringy black hole entropy (where the a_{ABD} are integers and the symmetry is $[SL(2,Z)]^3$) and three-qubit quantum entanglement (where the a_{ABD} are complex numbers and the symmetry is $[SL(2,C]^3$) is a purely mathematical coincidence. Nevertheless, it has already provided fascinating new insights [1, 2, 3, 4, 5] into the connections between strings, black holes and quantum information[1].

In Sect. 6 we extend the argument to the $N = 8$ case and, noting that

$$E_{7(7)}(Z) \supset [SL(2,Z)]^7 \tag{5}$$

and

$$E_7(C) \supset [SL(2,C)]^7, \tag{6}$$

show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they undergo at most tripartite entanglement of a very specific kind. As discussed in Sect. 8, the entanglement measure will be given by the quartic Cartan $E_7(C)$ invariant [16, 17, 18, 19]. The entanglement may be represented by the Fano plane [20] which also provides the multiplication table of the octonions. See also the interesting paper by Levay [5] who noted independently the connection to the Fano plane.

2 The N = 2 STU Model

Consider the three complex scalars axion/dilaton field S, the complex Kahler form field T and the complex structure field U

$$\begin{aligned} S &= S_1 + iS_2 \\ T &= T_1 + iT_2 \\ U &= U_1 + iU_2. \end{aligned} \tag{7}$$

This complex parameterization allows for a natural transformation under the various $SL(2,Z)$ symmetries. The action of $SL(2,Z)_S$ is given by

$$S \to \frac{aS+b}{cS+d}, \tag{8}$$

where a,b,c,d are integers satisfying $ad - bc = 1$, with similar expressions for $SL(2,Z)_T$ and $SL(2,Z)_U$. Defining the matrices $\mathscr{M}_S$, $\mathscr{M}_T$ and $\mathscr{M}_U$ via

$$\mathscr{M}_S = \frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix}, \tag{9}$$

the action of $SL(2,Z)_S$ now takes the form

$$\mathscr{M}_S \to {\omega_S}^T \mathscr{M}_S \omega_S, \tag{10}$$

[1] A third application [15], not considered in this paper, is the Nambu-Goto string whose action is also given by $\sqrt{|\mathrm{Det}\, a_{ABD}|}$ in spacetime signature (2,2).

where

$$\omega_S = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \tag{11}$$

with similar expressions for $\mathcal{M}_T$ and $\mathcal{M}_U$. We also define the $SL(2,Z)$ invariant tensors

$$\epsilon_S = \epsilon_T = \epsilon_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

Starting from the heterotic string, the bosonic action for the graviton $g_{\mu\nu}$, dilaton η, two-form $B_{\mu\nu}$ four $U(1)$ gauge fields A^a_S and two complex scalars T and U is [9]

$$\begin{aligned} I_{STU} = \ & \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-\eta}\bigg[R_g + g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta - \frac{1}{12}g^{\mu\lambda}g^{\nu\tau}g^{\rho\sigma}H_{\mu\nu\rho}H_{\lambda\tau\sigma} \\ & + \frac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}_T{}^{-1}\partial\mathcal{M}_T\right) + \frac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}_U{}^{-1}\partial\mathcal{M}_U\right) \\ & - \frac{1}{4}F_{S\mu\nu}{}^T(\mathcal{M}_T\times\mathcal{M}_U)F_S{}^{\mu\nu}\bigg]. \end{aligned} \tag{13}$$

where the metric $g_{\mu\nu}$ is related to the four-dimensional canonical Einstein metric $g^c_{\mu\nu}$ by $g_{\mu\nu} = e^\eta g^c{}_{\mu\nu}$ and where

$$H_{\mu\nu\rho} = 3\left(\partial_{[\mu}B_{\nu\rho]} - \tfrac{1}{2}A_{S[\mu}{}^T\left(\epsilon_T\times\epsilon_U\right)F_{S\nu\rho]}\right). \tag{14}$$

This action is manifestly invariant under T-duality and U-duality, with

$$F_{S\mu\nu} \to \left(\omega_T{}^{-1}\times\omega_U{}^{-1}\right)F_{S\mu\nu}, \qquad \mathcal{M}_{T/U} \to \omega^T_{T/U}\,\mathcal{M}_{T/U}\,\omega_{T/U}, \tag{15}$$

and with η, $g_{\mu\nu}$ and $B_{\mu\nu}$ inert. Its equations of motion and Bianchi identities (but not the action itself) are also invariant under S-duality (8), with T and $g^c{}_{\mu\nu}$ inert and with

$$\begin{pmatrix} F_{S\mu\nu}{}^a \\ \widetilde{F}_{S\mu\nu}{}^a \end{pmatrix} \to \omega_S^{-1}\begin{pmatrix} F_{S\mu\nu}{}^a \\ \widetilde{F}_{S\mu\nu}{}^a \end{pmatrix}, \tag{16}$$

where

$$\widetilde{F}_{S\mu\nu}{}^a = -S_2\left[\left(\mathcal{M}_T{}^{-1}\times\mathcal{M}_U{}^{-1}\right)\left(\epsilon_T\times\epsilon_U\right)\right]^a{}_b * F_{S\mu\nu}{}^b - S_1F_{S\mu\nu}{}^a, \tag{17}$$

where the axion field a is defined by

$$\epsilon^{\mu\nu\rho\sigma}\partial_\sigma a = \sqrt{-g}e^{-\eta}g^{\mu\sigma}g^{\nu\lambda}g^{\rho\tau}H_{\sigma\lambda\tau}, \tag{18}$$

and where $S = S_1 + iS_2 = a + ie^{-\eta}$.

Thus T-duality transforms Kaluza-Klein electric charges $(F_S{}^3,\ F_S{}^4)$ into winding electric charges $(F_S{}^1,\ F_S{}^2)$ (and Kaluza-Klein magnetic charges into winding magnetic charges), U-duality transforms the Kaluza-Klein and winding electric charge of one circle $(F_S{}^3,\ F_S{}^2)$ into those of the other $(F_S{}^4,\ F_S{}^1)$ (and similarly for the magnetic charges), but S-duality transforms Kaluza-Klein electric charge $({F_S}^3,\ {F_S}^4)$ into winding magnetic charge $(\tilde{F_S}^3,\ \tilde{F_S}^4)$ (and winding electric charge into Kaluza-Klein magnetic charge). In summary, we have $SL(2,\ Z)_T \times SL(2,\ Z)_U$ and $T \leftrightarrow U$ off-shell but $SL(2,\ Z)_S \times SL(2,\ Z)_T \times SL(2,\ Z)_U$ and an S–T–U interchange on-shell.

One may also consider the Type IIA action I_{TUS} and the Type IIB action I_{UST} obtained by cyclic permutation of the fields S,T,U. Finally, one may consider an action [11] where the S, T and U fields enter democratically with a prepotential

$$F = STU \tag{19}$$

which off-shell has the full STU interchange but none of the $SL(2,\ Z)$. All four versions are on-shell equivalent.

Following [9], it is now straightforward to write down an S–T–U symmetric Bogomolnyi mass formula. Let us define electric and magnetic charge vectors α_S^a and β_S^a associated with the field strengths ${F_S}^a$ and $\tilde{F}_S^a$ in the standard way. The electric and magnetic charges Q_S^a and P_S^a are given by

$$F_{S0r}^{\ a} \sim \frac{Q_S^a}{r^2} \qquad * F_{S0r}^{\ a} \sim \frac{P_S^a}{r^2}, \tag{20}$$

giving rise to the charge vectors

$$\begin{pmatrix} \alpha_S^a \\ \beta_S^a \end{pmatrix} = \begin{pmatrix} S_2^{(0)} \mathscr{M}_T^{-1} \times \mathscr{M}_U^{-1} & S_1^{(0)} \epsilon_T \times \epsilon_U \\ 0 & -\epsilon_T \times \epsilon_U \end{pmatrix}^{ab} \begin{pmatrix} Q_S^b \\ P_S^b \end{pmatrix}. \tag{21}$$

For our purpose it is useful to define a $2 \times 2 \times 2$ array a_{ABD} via

$$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} = \begin{pmatrix} -\beta_S^1 \\ -\beta_S^2 \\ -\beta_S^3 \\ -\beta_S^4 \\ \alpha_S^1 \\ \alpha_S^2 \\ \alpha_S^3 \\ \alpha_S^4 \end{pmatrix}, \tag{22}$$

transforming as

$$a^{ABD} \to {\omega_S}^A{}_{A'}\,{\omega_T}^B{}_{B'}\,{\omega_U}^D{}_{D'}\,a^{A'B'D'}. \tag{23}$$

Then the mass formula is

$$m^2 = \frac{1}{16} a^T \left({\mathscr{M}_S}^{-1} {\mathscr{M}_T}^{-1} {\mathscr{M}_U}^{-1} - {\mathscr{M}_S}^{-1} \epsilon_T \epsilon_U - \epsilon_S {\mathscr{M}_T}^{-1} \epsilon_U - \epsilon_S \epsilon_T {\mathscr{M}_U}^{-1} \right) a. \tag{24}$$

This is consistent with the general $N = 2$ Bogomolnyi formula [21]. Although all theories have the same mass spectrum, there is clearly a difference of interpretation with electrically charged elementary states in one picture being solitonic monopole or dyon states in the other.

This $2 \times 2 \times 2$ array a_{ABD} is an example of a "hypermatrix", a term coined by Cayley in 1845 [10], where he also introduced a "hyperdeterminant".

3 Cayley's Hyperdeterminant

In 1845, Cayley [10] generalized the determinant of a 2×2 matrix a_{AB} to the *hyperdeterminant* of a $2 \times 2 \times 2$ *hypermatrix* a_{ABD}

$$\begin{aligned}
\text{Det}\, a = & -\frac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4} \\
= & \, a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 \\
& - 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\
& + a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\
& + a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\
& + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}) \qquad (25) \\
= & \, a_0^2a_7^2 + a_1^2a_6^2 + a_2^2a_5^2 + a_3^2a_4^2 \\
& - 2(a_0a_1a_6a_7 + a_0a_2a_5a_7 + a_0a_4a_3a_7 + a_1a_2a_5a_6 + a_1a_3a_4a_6 + a_2a_3a_4a_5) \\
& + 4(a_0a_3a_5a_6 + a_1a_2a_4a_7) \qquad (26)
\end{aligned}$$

where we have made the binary conversion 0, 1, 2, 3, 4, 5, 6, 7 for 000, 001, 010, 011, 100, 101, 110, 111.

The hyperdeterminant vanishes iff the following system of equations in six unknowns p^A, q^B, r^D has a nontrivial solution, not allowing any of the pairs to be both zero:

$$\begin{aligned}
a_{ABD}p^Aq^B &= 0 \\
a_{ABD}p^Ar^D &= 0 \\
a_{ABD}q^Br^D &= 0 \qquad (27)
\end{aligned}$$

For our purposes, the important properties of the hyperdeterminant are that it is a quartic invariant under $[SL(2)]^3$ and under a triality that interchanges *A, B* and *D*. These properties are valid whether the a_{ABD} are complex, real or integer.

One way to understand this triality is to think of having three different metrics (Alice, Bob and Daisy)

$$\begin{aligned}
(\gamma_A)_{A_1A_2} &= \epsilon^{B_1B_2}\epsilon^{D_1D_2}a_{A_1B_1D_1}a_{A_2B_2D_2} \\
(\gamma_B)_{B_1B_2} &= \epsilon^{D_1D_2}\epsilon^{A_1A_2}a_{A_1B_1D_1}a_{A_2B_2D_2} \\
(\gamma_D)_{D_1D_2} &= \epsilon^{A_1A_2}\epsilon^{B_1B_2}a_{A_1B_1D_1}a_{A_2B_2D_2} \qquad (28)
\end{aligned}$$

Explicitly,

$$\gamma = \begin{pmatrix} 2(a_0a_6 - a_2a_4) & a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 \\ a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 & 2(a_1a_7 - a_3a_5) \end{pmatrix} \tag{29}$$

$$\beta = \begin{pmatrix} 2(a_0a_3 - a_1a_2) & a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 \\ a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 & 2(a_4a_7 - a_5a_6) \end{pmatrix} \tag{30}$$

$$\alpha = \begin{pmatrix} 2(a_0a_5 - a_4a_1) & a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 \\ a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 & 2(a_2a_7 - a_6a_3) \end{pmatrix} \tag{31}$$

All are equivalent, however, since

$$det\ \alpha = det\ \beta = det\ \gamma = -\mathrm{Det}\, a \tag{32}$$

If we make the identifications

$$\begin{aligned} a_0 &= \frac{1}{\sqrt{2}}\left(-P^0 + P^2\right) \\ a_1 &= \frac{1}{\sqrt{2}}\left(-Q^0 + Q^2\right) \\ a_2 &= \frac{1}{\sqrt{2}}\left(P^1 - P^3\right) \\ a_3 &= \frac{1}{\sqrt{2}}\left(Q^1 - Q^3\right) \\ a_4 &= \frac{1}{\sqrt{2}}\left(-P^1 - P^3\right) \\ a_5 &= \frac{1}{\sqrt{2}}\left(-Q^1 - Q^3\right) \\ a_6 &= \frac{1}{\sqrt{2}}\left(-P^0 - P^2\right) \\ a_7 &= \frac{1}{\sqrt{2}}\left(-Q^0 - Q^2\right) \end{aligned} \tag{33}$$

then we find the $O(2,2)$ scalar products

$$\begin{aligned} 2\left(a_0a_6 - a_2a_4\right) &= \left(P^0\right)^2 + \left(P^1\right)^2 - \left(P^2\right)^2 - \left(P^3\right)^2 = P^2 \\ 2\left(a_1a_7 - a_3a_5\right) &= (Q_0)^2 + (Q_1)^2 - (Q_2)^2 - (Q_3)^2 = Q^2 \\ a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 &= \left(P^0Q_0\right) + \left(P^1Q_1\right) + \left(P^2Q_2\right) + \left(P^3Q_3\right) = P.Q \end{aligned}$$

so

$$\gamma = \begin{pmatrix} P^2 & P.Q \\ P.Q & Q^2 \end{pmatrix} \tag{34}$$

and

$$-Det\ a = P^2Q^2 - (P.Q)^2$$

4 Black Hole Entropy

The *STU* model admits extremal black hole solutions satisfying the Bogomolnyi mass formula. As usual, their entropy is given by one quarter the area of the event horizon. However, to calculate this area requires evaluating the mass, not with the asymptotic values of the moduli but with their frozen values on the horizon, which are fixed in terms of the charges [22]. This ensures that the entropy is moduli-independent, as it should be. The relevant calculation was carried out in [11] for the model with the *STU* prepotential. The electric and magnetic charges of that paper are denoted $(p^0,\ q_0)$, $(p^1,\ q_1)$, $(p^2,\ q_2)$, $(p^3,\ q_3)$. In these variables, the entropy is given by

$$S = \pi \left(W(p^\Lambda, q_\Lambda) \right)^{1/2} \tag{35}$$

where

$$\begin{aligned} W\left(p^\Lambda, q_\Lambda\right) = & -(p\cdot q)^2 + 4\left(\left(p^1q_1\right)\left(p^2q_2\right) + \left(p^1q_1\right)\left(p^3q_3\right) + \left(p^3q_3\right)\left(p^2q_2\right)\right) \\ & - 4p^0q_1q_2q_3 + 4q_0p^1p^2p^3. \end{aligned} \tag{36}$$

The function $W(p^\Lambda,\ q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. For the solution to be BPS, we have to require $W > 0$.

If we make the identifications [1]

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -a_0 \\ -a_1 \\ -a_2 \\ a_4 \\ -a_7 \\ a_6 \\ a_5 \\ -a_3 \end{bmatrix} \tag{37}$$

we recognize from (3) that

$$W = -\text{Det}\, a, \tag{38}$$

and hence the black hole entropy is given by

$$S = \pi\sqrt{-\text{Det}\ a} \tag{39}$$

Some examples of supersymmetric black hole solutions [23] are provided by the electric Kaluza-Klein black hole with $\alpha = (1,0,0,0)$ and $\beta = (0,0,0,0)$; the electric winding black hole with $\alpha = (0,0,0,-1)$ and $\beta = (0,0,0,0)$; the magnetic Kaluza-Klein black hole with $\alpha = (0,0,0,0)$ and $\beta = (0,-1,0,0)$; the magnetic winding black hole with $\alpha = (0,0,0,0)$ and $\beta = (0,0,-1,0)$. These are characterized by a scalar-Maxwell coupling parameter $a = \sqrt{3}$. By combining these 1-particle states, we may build up 2-, 3- and 4-particle bound states at threshold [9, 23]. For example $\alpha = (1,0,0,-1)$ and $\beta = (0,0,0,0)$ with $a = 1$; $\alpha = (1,0,0,-1)$ and $\beta = (0,-1,0,0)$ with $a = 1/\sqrt{3}$; $\alpha = (1,0,0,-1)$ and $\beta = (0,-1,-1,0)$ with $a = 0$. The 1-, 2- and 3-particle states all yield vanishing contributions to Det a. A non-zero value is obtained for the 4-particle example, however, which is just the Reissner-Nordstrom black hole.

5 The N = 8 Generalization

The black holes described by Cayley's hyperdeterminant are those of $N = 2$ supergravity coupled to three vector multiplets, where the symmetry is $[SL(2,Z)]^3$. One might therefore ask whether the black hole/information theory correspondence could be generalized. There are three generalizations we might consider:

(1) $N = 2$ supergravity coupled to l vector multiplets where the symmetry is $SL(2,Z) \times SO(l-1,2,\ Z)$, and the black holes carry charges belonging to the $(2,l+1)$ representation ($l+1$ electric plus $l+1$ magnetic).
(2) $N = 4$ supergravity coupled to m vector multiplets where the symmetry is $SL(2,Z) \times SO(6,6+m,\ Z)$, where the black holes carry charges belonging to the $(2,12+m)$ representation ($m+12$ electric plus $m+12$ magnetic).
(3) $N = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{7(7)}(Z)$, and the black holes carry charges belonging to the fundamental 56-dimensional representation (28 electric plus 28 magnetic).

In all three cases, there exist quartic invariants akin to Cayley's hyperdeterminant whose square root yields the corresponding black hole entropy. If there is to be a quantum information theoretic interpretation, however, it cannot just be random entanglement of more qubits, because the general n qubit entanglement is described by the group $[SL(2,C)]^n$, which, even after replacing Z by C, differs from the above symmetries (except when $n = 3$, which correspond to case (1) above with $l = 3$, the case we already know).

We note, however, that

$$E_{7(7)}(Z) \supset [SL(2,Z)]^7 \tag{40}$$

and

$$E_7(C) \supset [SL(2,C)]^7, \tag{41}$$

We shall now show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they undergo at most tripartite entanglement of a very specific kind. The entanglement measure will be given by the quartic Cartan $E_7(C)$ invariant [16, 17, 18, 19].

6 Decomposition of $E_{7(7)}$

Consider the decomposition of the fundamental 56-dimensional representation of $E_{7(7)}$ under its maximal subgroup

$$\begin{aligned} E_{7(7)} &\supset SL(2)_A \times SO(6,6) \\ 56 &\to (2,12)+(1,32) \end{aligned} \tag{42}$$

Further decomposing $SO(6,6)$,

$$\begin{aligned} SL(2)_A \times SO(6,6) &\supset SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(4,4) \\ (2,12)+(1,32) &\to (2,2,2,1) \\ &+(2,1,1,8_v)+(1,2,1,8_s)+(1,1,2,8_c) \end{aligned} \tag{43}$$

Further decomposing $SO(4,4)$,

$$\begin{aligned} SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(4,4) &\supset SL(2)_A \times SL(2)_B \times SL(2)_D \\ &\times SO(2,2) \times SO(2,2) \\ (2,2,2,1)+(2,1,1,8_v)+(1,2,1,8_s)+(1,1,2,8_c) &\to \\ (2,2,2,1,1)+(2,1,1,4,1)+(2,1,1,1,4) \\ +(1,2,1,2,2)+(1,2,1,2,2)+(1,1,2,2,2)+(1,1,2,2,2) \end{aligned} \tag{44}$$

Finally, further decomposing each $SO(2,2)$

$$\begin{aligned} SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(2,2) \times SO(2,2) \supset \\ SL(2)_A \times SL(2)_B \times SL(2)_D \times SL(2)_C \times SL(2)_G \times SL(2)_F \times SL(2)_E \\ (2,2,2,1,1)+(2,1,1,4,1)+(2,1,1,1,4) \\ +(1,2,1,2,2)+(1,2,1,2,2)+(1,1,2,2,2)+(1,1,2,2,2) \to \\ (2,2,2,1,1,1,1)+(2,1,1,2,2,1,1)+(2,1,1,1,1,2,2)+ \\ (1,2,1,2,1,1,2)+(1,2,1,1,2,2,1)+(1,1,2,2,1,2,1)+(1,1,2,1,2,1,2) \end{aligned}$$

In summary,

$$E_{7(7)} \supset SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F \times SL(2)_G \quad (45)$$

and the 56 decomposes as

$$\begin{aligned} &56 \rightarrow \\ &(2,2,1,2,1,1,1) \\ &+(1,2,2,1,2,1,1) \\ &+(1,1,2,2,1,2,1) \\ &+(1,1,1,2,2,1,2) \\ &+(2,1,1,1,2,2,1) \\ &+(1,2,1,1,1,2,2) \\ &+(2,1,2,1,1,1,2) \end{aligned} \quad (46)$$

An analogous decomposition holds for

$$E_7(C) \supset [SL(2,C)]^7. \quad (47)$$

7 Tripartite Entanglement of 7 Qubits

We have seen that in the case of three qubits the tripartite entanglement is described by $[SL(2,C)]^3$, and that the entanglement measure is given by Cayley's hyperdeterminant. Now we consider seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) but where Alice has tripartite entanglement not only with Bob/Daisy but also with Emma/Fred and also with George/Charlie, and similarly for the other six individuals. So, in fact, each person has tripartite entanglement with each of the remaining three couples:

$$\begin{aligned} &|\Psi\rangle = \\ &a_{ABD}|ABD\rangle \\ &+b_{BCE}|BCE\rangle \\ &+c_{CDF}|CDF\rangle \\ &+d_{DEG}|DEG\rangle \\ &+e_{EFA}|EFA\rangle \\ &+f_{FGB}|FGB\rangle \\ &+g_{GAC}|GAC\rangle \end{aligned} \quad (48)$$

Note that

(1) Any pair of states has an individual in common
(2) Each individual is excluded from four out of the seven states
(3) Two given individuals are excluded from two out of the seven states
(4) Three given individuals are never excluded

The entanglement may be represented by a heptagon with vertices A,B,C,D,E,F,G and seven triangles ABD, BCE, CDF, DEG, EFA, FGB and GAC. See Fig. 1. Alternatively, we can use the Fano plane. See Fig. 2. The Fano plane corresponds to the multiplication table of the octonions as may be seen from the description of the state $|\Psi\rangle$ given in Table 1.

Each of the seven states transforms as a (2,2,2) under three of the $SL(2)$s and are singlets under the remaining four. Note that from (43) we see that the A-B-C triality of Sect. 3 is linked with the $8_v - 8_s - 8_c$ triality of the $SO(4,4)$. For example, interchanging A and B leaves $|\Psi\rangle$ invariant provided we also interchange C and F. Individually, therefore, the tripartite entanglement of each of the seven states is

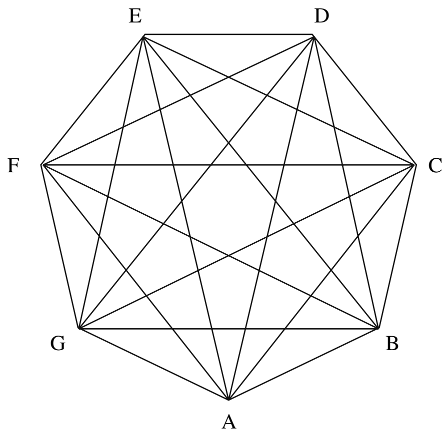

Fig. 1 The E_7 entanglement diagram. Each of the seven vertices A,B,C,D,E,F,G represents a qubit, and each of the seven triangles ABD, BCE, CDF, DEG, EFA, FGB, GAC describes a tripartite entanglement

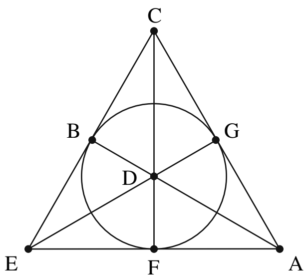

Fig. 2 The Fano plane has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point

Table 1 The entanglement of the state $|\Psi\rangle$ coincides with the multiplication table of the octonions

	A	B	C	D	E	F	G
A		D	G	−B	F	−E	−C
B	−D		E	A	−C	G	−F
C	−G	−E		F	B	−D	A
D	B	−A	−F		G	C	−E
E	−F	C	−B	−G		A	D
F	E	−G	D	−C	−A		B
G	C	F	−A	E	−D	−B	

given by Cayley's hyperdeterminant. Taken together, however, we see from (46) that they transform as a complex 56 of $E_7(C)$. Their tripartite entanglement must be given by an expression that is quartic in the coefficients *a,b,c,d,e,f,g* and invariant under $E_7(C)$. The unique possibility is the Cartan invariant I_4, and so the 3-tangle is given by

$$\tau_3(ABCDEFG) = 4|I_4| \tag{49}$$

If the wave-function (47) is normalized, then $0 \leq \tau_3(ABCDEFG) \leq 1$.

8 Cartan's $E_{7(7)}$ Invariant

The Cremmer-Julia [17] form of the Cartan $E_{7(7)}$ invariant may be written as

$$I_4 = \mathrm{T}r\,(Z\bar{Z})^2 - \frac{1}{4}\,(\mathrm{T}r\,Z\bar{Z})^2 + 4(\mathrm{P}f\,Z + \mathrm{P}f\,\bar{Z}) \tag{50}$$

and the Cartan form [16] may be written as

$$I_4 = -\mathrm{T}r\,(x\,y)^2 + \frac{1}{4}(\mathrm{T}r\,x\,y)^2 - 4(\mathrm{P}f\,x + \mathrm{P}f\,y). \tag{51}$$

Here

$$Z_{AB} = -\frac{1}{4\sqrt{2}}(x^{ab} + iy_{ab})(\Gamma^{ab})_{AB} \tag{52}$$

and

$$x^{ab} + iy_{ab} = -\frac{\sqrt{2}}{4} Z_{AB}(\Gamma^{AB})_{ab} \tag{53}$$

The matrices of the $SO(8)$ algebra are $(\Gamma^{ab})_{AB}$, where $(a\ b)$ are the 8 vector indices and $(A,\ B)$ are the 8 spinor indices. The $(\Gamma^{ab})_{AB}$ matrices can be considered also as $(\Gamma^{AB})_{ab}$ matrices due to equivalence of the vector and spinor representations of the $SO(8)$ Lie algebra. The exact relation between the Cartan invariant in (51) and Cremmer-Julia invariant [17] in (50) was established in [24, 25]. The quartic invariant I_4 of $E_{7(7)}$ is also related to the octonionic Jordan algebra J_3^O [19].

In the stringy black hole context, Z_{AB} is the central charge matrix, and (x,y) are the quantized charges of the black hole (28 electric and 28 magnetic). The relation between the entropy of stringy black holes and the Cartan-Cremmer-Julia $E_{7(7)}$ invariant was established in [18]. The central charge matrix Z_{AB} can be brought to the canonical basis for the skew-symmetric matrix using an $SU(8)$ transformation:

$$Z_{ab} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{54}$$

where $z_i = \rho_i e^{i\varphi_i}$ are complex. In this way, the number of entries is reduced from 56 to 8. In a systematic treatment in [26], the meaning of these parameters was clarified. From 4 complex values of $z_i = \rho_i e^{i\varphi_i}$ one can remove 3 phases by an $SU(8)$ rotation, but the overall phase cannot be removed; it is related to an extra parameter in the class of black hole solutions [27, 28, 29]. In this basis, the quartic invariant takes the form [18]

$$\begin{aligned} I_4 &= \sum_i |z_i|^4 - 2\sum_{i<j} |z_i|^2 |z_j|^2 + 4(z_1 z_2 z_3 z_4 + \bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4) \\ &= (\rho_1+\rho_2+\rho_3+\rho_4)(\rho_1+\rho_2-\rho_3-\rho_4)(\rho_1-\rho_2+\rho_3-\rho_4) \\ &\quad \times (\rho_1-\rho_2-\rho_3+\rho_4) + 8\rho_1\rho_2\rho_3\rho_4(\cos\varphi - 1) \end{aligned} \tag{55}$$

Therefore a 5-parameter solution is called a generating solution for other black holes in $\mathrm{N} = 8$ supergravity/M-theory. The expression for their entropy is always given by

$$S = \pi\sqrt{|I_4|} \tag{56}$$

for some subset of 5 of the 8 parameters mentioned above. Recently a new class of solutions was discovered, describing black rings. The maximal number of parameters for the known solutions is 7. The entropy of black ring solutions found so far was identified in [30, 31] with the expression (56) for a subset of 7 out of 8 parameters mentioned above.

Kallosh and Linde have shown that I_4 depending on 4 complex eigenvalues can be represented as Cayley's hyperdeterminant of a hypermatrix a_{ABD}. To see this, we note that in $x,\ y$ basis only the $SO(8)$ symmetry is manifest, which means that every term in (51) is invariant only under $SO(8)$ symmetry. However, it was proved in [16] and [17] that the sum of all terms in (51) is invariant under the full $SU(8)$ symmetry, which acts as follows

$$\delta\left(x^{ab} \pm i y_{ab}\right) = \left(2\Lambda^{[a}{}_{[c}\delta^{b]}{}_{d]} \pm i\Sigma_{abcd}\right)\left(x^{cd} \mp i y_{cd}\right). \tag{57}$$

The total number of parameters is 63, where 28 are from the manifest $SO(8)$ and 35 from the antisymmetric self-dual $\Sigma_{abcd} =^* \Sigma^{abcd}$. Thus one can use the $SU(8)$ transformation of the complex matrix $x^{ab} + i y_{ab}$ and bring it to the canonical form with some complex eigenvalues λ_I, $I = 1,\ 2,\ 3,\ 4$. The value of the quartic invariant (51) will not change.

$$\left(x^{ab} + iy_{ab}\right)_{\text{can}} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_4 & 0 \end{pmatrix} \tag{58}$$

The relation between the complex coefficients λ_I, the parameters x^{ij} and y_{kl}, the matrix a_{ABD} and the black hole charges p^i and q_k [1] is given by the following dictionary:

$$\begin{aligned} \lambda_1 &= x^{12} + iy_{12} = a_{111} + ia_{000} = -q_0 - ip^0 \\ \lambda_2 &= x^{34} + iy_{34} = a_{001} + ia_{110} = -p^1 + iq_1 \\ \lambda_3 &= x^{56} + iy_{56} = a_{010} + ia_{101} = -p^2 + iq_2 \\ \lambda_4 &= x^{78} + iy_{78} = a_{100} + ia_{011} = p^3 - iq_3 \end{aligned} \tag{59}$$

If we now write the quartic $E_{7(7)}$ Cartan invariant in the canonical basis (x^{ij}, y_{ij}), $i, j = 1, \ldots, 8$:

$$\begin{aligned} I_4 = &- \left(x^{12}y_{12} + x^{34}y_{34} + x^{56}y_{56} + x^{78}y_{78}\right)^2 - 4\left(x^{12}x^{34}x^{56}x^{78} + y_{12}y_{34}y_{56}y_{78}\right) \\ &+ 4\left(x^{12}x^{34}y_{12}y_{34} + x^{12}x^{56}y_{12}y_{56} + x^{34}x^{56}y_{34}y_{56} + x^{12}x^{78}y_{12}y_{78} + x^{34}x^{78}y_{34}y_{78} \right. \\ &\left. + x^{56}x^{78}y_{56}y_{78}\right). \end{aligned} \tag{60}$$

then it may now be compared to Cayley's hyperdeterminant (25). We find

$$I_4 = -\text{Det}\, a \tag{61}$$

The above discussion of $E_{7(7)}$ also applies, mutatis mutandis, to $E_7(C)$.

To understand better the entanglement, we note that as a result of (46) Cartan's invariant contains not one Cayley hyperdeterminant but seven! It may be written as the sum of seven terms each of which is invariant under $[SL(2)]^3$ plus cross terms. To see this, denote a 2 in one of the seven entries in (46) by A, B, C, D, E, F, G. So we may rewrite (46) as

$$56 = (ABD) + (BCE) + (CDF) + (DEG) + (EFA) + (FGB) + (GAC) \tag{62}$$

or symbolically

$$56 = a + b + c + d + e + f + g \tag{63}$$

Then I_4 is the singlet in $56 \times 56 \times 56 \times 56$:

$$\begin{aligned}J_4 \sim a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 + \\ 2\left[a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2f^2 + f^2g^2 + g^2a^2 + \right. \\ a^2c^2 + b^2d^2 + c^2e^2 + d^2f^2 + e^2g^2 + f^2a^2 + g^2b^2 + \\ \left. a^2d^2 + b^2e^2 + c^2f^2 + d^2g^2 + e^2a^2 + f^2b^2 + g^2c^2\right] \\ +8[bcdf + cdeg + defa + efgb + fgac + gabd + abce]\end{aligned} \tag{64}$$

where products like

$$\begin{aligned}a^4 &= (ABD)(ABD)(ABD)(ABD) \\ &= \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4}\end{aligned} \tag{65}$$

exclude four individuals (here Charlie, Emma, Fred and George), products like

$$\begin{aligned}a^2f^2 &= (ABD)(ABD)(FGB)(FGB) \\ &= \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{F_3F_4}\epsilon^{G_3G_4}\epsilon^{D_2B_3}a_{A_1B_1D_1}a_{A_2B_2D_2}f_{F_3G_3B_3}f_{F_4G_4B_4}\end{aligned} \tag{66}$$

exclude two individuals (here Charlie and Emma) and products like

$$\begin{aligned}abce &= (ABD)(BCE)(CDF)(EFA) \\ &= \epsilon^{A_1B_2}\epsilon^{B_1C_2}\epsilon^{D_1A_4}\epsilon^{C_3E_4}\epsilon^{D_3F_4}\epsilon^{E_2F_3}a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4}\end{aligned} \tag{67}$$

exclude one individual (here George).

9 The Black Hole Analogy

In the STU stringy black hole context [1, 2, 9, 11], the a_{ABC} are integers (corresponding to quantized charges), and hence the symmetry group is $[SL(2,Z)]^3$ rather than $[SL(2,C)]^3$. However, as discussed by Levay [3], there is a branch of quantum information theory which concerns itself with real qubits, called *rebits*, for which the a_{ABC} are real. (One difference remains, however: one may normalize the wave function, whereas for black holes there is no such restriction on the charges a_{ABC}.) It turns out that there are three reality classes which can be characterized by the hyperdeterminant

$$\begin{aligned}&1)\ \text{Det}\, a < 0 \\ &2)\ \text{Det}\, a = 0 \\ &3)\ \text{Det}\, a > 0\end{aligned} \tag{68}$$

Case (1) corresponds to the non-separable or GHZ class [14], for example,

$$|\Psi\rangle = \frac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{69}$$

Case (2) corresponds to the separable (A-B-C, A-BC, B-CA, C-AB) and W classes, for example

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{70}$$

In the string/supergravity interpretation [1], cases (1) and (2) were shown to correspond to BPS black holes, for which half of the supersymmetry is preserved. Case (1) has non-zero horizon area and entropy ("large" black holes), and case (2) to vanishing horizon area and entropy ("small" black holes), at least at the semi-classical level. However, small black holes may acquire a non-zero entropy through higher-order quantum effects. This entropy also has a quantum information interpretation involving bipartite entanglement of the three qubits [2].

Case (3) is also GHZ, for example the above GHZ state (69) with a sign flip

$$|\Psi\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{71}$$

In the string/supergravity interpretation, case (3) corresponds to non-BPS black holes [2]. With four non-zero charges (q_0, p^1, p^2, p^3) in (59), for example, an extreme but non-BPS black hole [23] may be obtained by flipping the sign [32] of one of the charges. The canonical GHZ state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|111\rangle + \frac{1}{\sqrt{2}}|000\rangle \tag{72}$$

also belongs to case (3).

In the $N = 8$ theory, "large" and "small" black holes are classified by the sign of I_4:

$$\begin{aligned} &1)\ I_4 > 0 \\ &2)\ I_4 = 0 \\ &3)\ I_4 < 0 \end{aligned} \tag{73}$$

Once again, non-zero I_4 corresponds to large black holes, which are BPS for $I_4 > 0$ and non-BPS for $I_4 < 0$, and vanishing I_4 to small black holes. However, in contrast to $N = 2$, case (1) requires that only 1/8 of the supersymmetry is preserved, while we may have 1/8, 1/4 or 1/2 for case (2).

It is worth noting that the charge orbits corresponding to non-zero I_4 are associated with the following cosets:

$$\frac{E_{7(7)}}{E_{6(2)}} \tag{74}$$

and

$$\frac{E_{7(7)}}{E_{6(6)}} \tag{75}$$

The large black hole solutions can be found [33] by solving the $N = 8$ classical attractor equations [22] when at the attractor value the Z_{AB} matrix, in normal form, becomes

$$Z_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{76}$$

for positive I_4 and

$$Z_{AB} = e^{i\pi/4}|Z| \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{77}$$

for negative I_4. These values exhibit the maximal compact symmetries $SU(6) \times SU(2)$ and $USp(8)$ for the positive and negative I_4, respectively.

If the phase in (55) vanishes (which is the case if the configuration preserves at least 1/4 supersymmetry [26]), I_4 becomes

$$I_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4, \tag{78}$$

where we have defined λ_i by

$$\begin{aligned} \lambda_1 &= \rho_1 + \rho_2 + \rho_3 + \rho_4 \\ \lambda_2 &= \rho_1 + \rho_2 - \rho_3 - \rho_4 \\ \lambda_3 &= \rho_1 - \rho_2 + \rho_3 - \rho_4 \\ \lambda_4 &= \rho_1 - \rho_2 - \rho_3 + \rho_4 \end{aligned} \tag{79}$$

and we order the λ_i so that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq |\lambda_4|$. The charge orbits for the small black holes depend on the number of unbroken supersymmetries or the number of vanishing eigenvalues. The orbit is [19, 26, 34]

$$\frac{E_{7(7)}}{H_{1,2,3}} \tag{80}$$

where

$$\begin{aligned} &H_1 = F_{4(4)} \ltimes T_{26} && \lambda_1,\ \lambda_2,\ \lambda_3 \neq 0,\ \lambda_4 = 0 && (1/8\ BPS) \\ &H_2 = SO(5,6) \ltimes (T_{32} \times T_1) && \lambda_1,\ \lambda_2 \neq 0,\ \lambda_3,\ \lambda_4 = 0 && (1/4\ BPS) \\ &H_3 = E_{6(6)} \ltimes T_{27} && \lambda_1 \neq 0,\ \lambda_2,\ \lambda_3,\ \lambda_4 = 0 && (1/2\ BPS) \end{aligned} \tag{81}$$

For $N = 8$, as for $N = 2$, the large black holes correspond to the two classes of GHZ-type (entangled) states and small black holes to the separable or W class.

10 Subsectors

Having understood the analogy between $N = 8$ black holes and the tripartite entanglement of 7 qubits using $E_{7(7)}$, we may now find the analogy in the $N = 4$ case using $SL(2) \times SO(6,6)$ and the $N = 2$ case using $SL(2) \times SO(2,2)$.

For $N = 4$, as may be seen from (43), we still have an $[SL(2)]^7$ subgroup but now there are only 24 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle \tag{82}$$

So only Alice talks to all the others. This is described by just those three lines passing through A in the Fano plane. Then the equations analagous to (62) and (63) are

$$(2,12) = (ABD) + (EFA) + (GAC) = a + e + g \tag{83}$$

and the corresponding quartic invariant, I_4, reduces to the singlet in $(2,12) \times (2,12) \times (2,12) \times (2,12)$.

$$I_4 \sim a^4 + e^4 + g^4 + 2\left[e^2g^2 + g^2a^2 + a^2e^2\right] \tag{84}$$

If we identify the 24 numbers $(a_{ABD}, e_{EFA}, g_{GAC})$ with (P^μ, Q_ν) with $\mu, \nu = 0, \ldots 11$, this becomes [9, 27, 28, 29]

$$I_4 = P^2Q^2 - (P.Q)^2 \tag{85}$$

which is manifestly invarinat under $SL(2) \times SO(6,6)$.

For $N = 2$, as may be seen from (43), we only have an $[SL(2)]^3$ subgroup and there are only 8 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle \tag{86}$$

This is described by just the ABD line in the Fano plane. This is simply the usual tripartite entanglement, for which

$$(2,2,2) = (ABD) = a \tag{87}$$

and the corresponding quartic invariant

$$I_4 \sim a^4 \tag{88}$$

is just Cayley's hyperdeterminant

$$I_4 = -\mathrm{Det}a \tag{89}$$

11 Conclusions

We note that the 56-dimensional Hilbert space given in (46) and (48) is not a subspace of the usual 2^7-dimensional seven-qubit Hilbert space given by (2,2,2,2,2,2,2) but rather a direct sum of seven 2^3-dimensional three-qubit Hilbert spaces (2,2,2). This is, however, a subspace of the 3^7-dimensional seven-qutrit Hilbert space given by (3,3,3,3,3,3,3). Under

$$[SL(3)]^7 \to [SL(2)]^7 \tag{90}$$

we have the decomposition

$$\begin{aligned}
&(3,3,3,3,3,3,3) \to \\
&1 \text{ term like } (2,2,2,2,2,2,2) \\
&7 \text{ terms like } (2,2,2,2,2,2,1) \\
&21 \text{ terms like } (2,2,2,2,2,1,1) \\
&35 \text{ terms like } (2,2,2,2,1,1,1) \\
&35 \text{ terms like } (2,2,2,1,1,1,1) \\
&21 \text{ terms like } (2,2,1,1,1,1,1) \\
&7 \text{ terms like } (2,1,1,1,1,1,1) \\
&1 \text{ term like } (1,1,1,1,1,1,1)
\end{aligned} \tag{91}$$

which contains

$$\begin{aligned}
&(2,2,1,2,1,1,1) \\
&+(1,2,2,1,2,1,1) \\
&+(1,1,1,2,2,1,2) \\
&+(2,1,1,1,2,2,1) \\
&+(1,2,1,1,1,2,2) \\
&+(2,1,2,1,1,1,2)
\end{aligned} \tag{92}$$

So the Fano plane entanglement we have described fits within conventional quantum information theory.

The Fano plane also finds application in switching networks that can connect any phone to any other phone. It is the 3-switching network for 7 numbers. However there also exists a 4-switching network for 13 numbers, a 5-switching network for 21 numbers and generally an $(n+1)$-switching network for (n^2+n+1) numbers corresponding to the projective planes of order n [35]. It would be worthwhile pursuing the corresponding quantum bit entanglements.

Exceptional groups, such as $E_{7(7)}$, have featured in supergravity, string theory, M-theory and other speculative attempts at unification of the fundamental forces.

However, it is unusual to find an exceptional group appearing in the context of qubit entanglement. It would be interesting to see whether it can be subject to experimental test.

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Extremal Black Hole and Flux Vacua Attractors

S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani

Abstract These lectures provide a pedagogical, introductory review of the so-called Attractor Mechanism (AM) at work in two different 4-dimensional frameworks: extremal black holes in $\mathscr{N}=2$ supergravity and $\mathscr{N}=1$ flux compactifications. In the first case, AM determines the stabilization of scalars at the black hole event horizon purely in terms of the electric and magnetic charges, whereas in the second context, the AM is responsible for the stabilization of the universal axion-dilaton and of the (complex structure) moduli purely in terms of the RR and NSNS fluxes. Two equivalent approaches to AM, namely the so-called "criticality conditions" and "New Attractor" ones, are analyzed in detail in both frameworks, whose analogies and differences are discussed. Also, a stringy analysis of both frameworks (relying on Hodge-decomposition techniques) is performed, respectively, considering Type IIB compactified on CY_3 and its orientifolded version, associated with $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$. Finally, recent results on the U-duality orbits and moduli spaces of non-BPS extremal black hole attractors in $3 \leq \mathscr{N} \leq 8,\ d=4$ supergravities are reported.

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1 Introduction

After the original papers [1, 2, 3, 4, 5] from the mid 90s dealing mostly with the Bogomol'ny-Prasad-Sommerfeld (BPS) black holes (BHs), *extremal BH attractors* have been recently widely investigated [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53] (see also [54, 55, 56, 57, 58, 59, 60, 61, 62]). Such a *renaissance* is mainly due to the (re)discovery of new classes of solutions to the attractor equations corresponding to non-BPS horizon geometries: Certain configurations of moduli stabilized near the horizon of extremal BHs exist which break supersymmetry. In addition, the stabilization of moduli in the context of string theory has become a central issue of string cosmology. The attractor equations used in the past for stabilizing moduli near the horizon of an extremal BH have turned out to be useful in the context of flux vacua.

In this introduction, we will first briefly remind the basic structure of the BPS BH attractors in $\mathcal{N}=2,\ d=4$ supergravity. After that, we will outline the main features of the recent developments in non-BPS extremal BH attractors and flux vacua, a detailed description of which will be given in the subsequent sections.

A horizon extremal BH attractor geometry is in general supported by particular configurations of the $1\times(2n_V+2)$ symplectic vector of the BH field-strength fluxes, i.e. of the BH magnetic and electric charges:

$$Q\equiv\left(p^\Lambda,q_\Lambda\right),\qquad p^\Lambda\equiv\frac{1}{4\pi}\int_{S^2_\infty}\mathscr{F}^\Lambda,\quad q_\Lambda\equiv\frac{1}{4\pi}\int_{S^2_\infty}\mathscr{G}_\Lambda,\quad \Lambda=0,1,\ldots,n_V,\tag{1}$$

where, in the case of $\mathcal{N}=2,\ d=4$ supergravity, n_V denotes the number of Abelian vector supermultiplets coupled to the supergravity one (containing the Maxwell vector A^0, usually named *graviphoton*). Here $\mathscr{F}^\Lambda=dA^\Lambda$ and $\mathscr{G}_\Lambda$ is the "dual" field-strength two-form [63, 64].

BPS BH attractor equations fix the values of all moduli near BH horizon in terms of the electric and magnetic charges. The most compact form of these equations was given in [65], where the Kähler invariant period $\left(Y^\Lambda,F_\Lambda(Y)\right)$ was introduced by multiplying the covariantly holomorphic period $V(z,\ \bar{z})$ (see (25) below) on the (complex conjugate of the) $\mathcal{N}=2,\ d=4$ *central charge function* $\bar{Z}$ so that

$$\bar{Z}V\equiv\left(Y^\Lambda\ ,F_\Lambda(Y)\right)\tag{2}$$

where $Y^\Lambda=Y^\Lambda(z,\ \bar{z})$ and $V=\left(L^\Lambda,\ M_\Lambda\right)$. In terms of such variables, the BPS attractor equations are very simple and state that at the BH horizon, the moduli $(z,\ \bar{z})$ depend on electric and magnetic charges so that equations

$$Y^\Lambda-\bar{Y}^\Lambda=ip^\Lambda,\qquad F_\Lambda(Y)-\bar{F}_\Lambda(\bar{Y})=iq_\Lambda\tag{3}$$

are satisfied, and their solution defines moduli as functions of charges

$$z_{cr}=z_{cr}(p,q)\ ,\qquad \bar{z}_{cr}=\bar{z}_{cr}(p,q).\tag{4}$$

BPS attractors equations (3) are equivalent to the condition of unbroken supersymmetry: $DZ = 0$.

A simple way to derive the BH attractor equations, which also gives a clear link to their use in the context of flux vacua, is by using the language of string theory compactified on a Calabi-Yau threefold (CY_3) [66, 67, 68]. One starts with the Hodge decomposition of the 3-form flux (see (189) below)

$$\mathscr{H}_3 = -2Im[\bar{Z}\hat{\Omega}_3 - \bar{D}^i\bar{Z}D_i\hat{\Omega}_3] = \int_{S^2_\infty} \widehat{\mathscr{F}}^+, \tag{5}$$

where $\hat{\Omega}_3$ is the covariantly holomorphic 3-form of the CY_3, $\widehat{\mathscr{F}}^+$ is the self-dual 5-form of type IIB string theory and S^2_∞ is the 2-sphere at infinity, as in the definition (1) (see e.g. [64]). By integration over a symplectic basis of 3-cycles of CY_3 the decomposition (5) can be brought to the form (see (140) below)

$$Q^T = -2Im[\bar{Z}V - \bar{D}^i\bar{Z}D_iV]. \tag{6}$$

By inserting the condition of unbroken supersymmetry $D_iZ = 0$ into the identities (5) and (6), one obtains the BPS extremal BH attractor equations (3) in a stringy framework:

$$\mathscr{H}_3 = -2Im[\bar{Z}\hat{\Omega}_3]_{DZ=0}, \tag{7}$$

or equivalently:

$$Q^T = -2Im\left[\overline{Z}V\right]_{DZ=0}. \tag{8}$$

This attractor equation presents a particular case of the criticality condition for the so-called effective BH potential, $\partial_i V_{BH} = 0$, where (see definition (48) below)

$$V_{BH}(z,\bar{z}) \equiv |Z|^2 + g^{i\bar{j}}(D_iZ)\overline{D}_{\bar{j}}\overline{Z}. \tag{9}$$

Another important feature of the BPS attractors is the relation between the second derivative of V_{BH} at the critical points $\partial V_{BH} = 0$ and the metric $g_{i\bar{j}}$ of the scalar manifold (usually called *moduli space* in string theory), namely

$$\left(\partial_i\overline{\partial}_{\bar{j}}V_{BH}\right)_{\partial V_{BH}=0} = 2\left(g_{i\bar{j}}V_{BH}\right)_{\partial V_{BH}=0}. \tag{10}$$

Since V_{BH} at the supersymmetric critical point $DZ = 0$ (with non-vanishing entropy) is strictly positive $\left(V_{BH}|_{DZ=0} = |Z|^2_{DZ=0} > 0\right)$, (10) implies that all BPS attractors are stable, at least as long as the metric of the moduli space is strictly positive definite. Note that in the BPS case the condition of non-vanishing entropy requires $Z|_{DZ=0} \neq 0$.

The recent developments with *non-BPS BH attractors* can be described shortly as follows. For extremal non-BPS BH solutions of $\mathcal{N} = 2$, $d = 4$ supergravity one finds the mechanism of stabilization of moduli near BH horizon with some properties of the same nature as in BPS case and some properties somewhat different.

Many nice features of the BH attractors in the past were associated with the unbroken supersymmetry of BPS BHs. During the last few years the basic reason for the attractor behaviour of extremal BHs has been discovered to be geometrical[1]: extremal BHs (regardless their supersymmetry-preserving features) all have moduli which acquire fixed values at the BH horizon independent of their values at infinity! Their values at the horizon depend only on the electric and magnetic BH charges. The existence of an infinite throat in the space-time geometry of extremal BHs leads to an evolution towards the horizon such that the moduli forget their initial conditions at (spatial) infinity [14]. Since a Schwarzschild-type BH geometry with non-vanishing horizon area is never extremal, this phenomenum never takes place when solving the equations of motion for scalar fields in such a background: Their values at the horizon depend on the initial conditions of the radial dynamical evolution, because there are no coordinate systems with infinite distance from the event horizon.

A simple qualifier of both BPS and non-BPS attractors remains valid in the form of a critical point of the BH potential:

$$\partial V_{BH} = 0 \ : \qquad for\ BPS: \quad DZ = 0, \qquad for\ non-BPS: \quad DZ \neq 0. \tag{11}$$

The non-BPS attractor equations in the form generalizing (3) can be given separately for the cases $Z \neq 0$ and $Z = 0$[2].

In the case $Z \neq 0$ one finds (see (157) below)

$$Q^T = -2Im\left\{\left[\overline{Z}V - \frac{i}{2}\frac{Z}{|Z|^2}C^{\overline{i}\overline{j}\overline{k}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{D}_{\overline{i}}\overline{V}\right]\right\}_{non-BPS, Z\neq 0}. \tag{12}$$

Here, one starts with the identity (6) and replaces the second term using the expression for it derived from the non-BPS $Z \neq 0$ criticality condition $\partial V_{BH} = 0$. The attractor equation (12) is a clear generalization of the BPS attractor equations (3), (4), (5), (6), (7), (8) with $Z \neq 0$: at $DZ = 0$ the second term in the right-hand side (r.h.s.) of (12) vanishes, and it reduces exactly to (8) or its detailed form given by (3).

For both classes ($Z \neq 0$ and $Z = 0$) of non-BPS attractors, the critical value of V_{BH} remains positive, since by definition V_{BH} is a real, positive function in the scalar manifold. However, the universal BPS stability condition (10) is not valid anymore and one has to study this issue separately[3].

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[1] We are grateful to A. Linde for this insight, see also [14].

[2] The non-BPS BH attractor equations $\partial V_{BH} = 0$ with the condition $Z = 0$, $DZ \neq 0$ will be discussed later in the lectures.

[3] In case that the critical Hessian matrix has some "massless modes" (i.e. vanishing eigenvalues), one has to look at higher-order covariant derivatives of V_{BH} evaluated at the considered point and study their sign. Depending on the configurations of the BH charges, one can obtain stable or unstable critical points.

Examples in literature of investigations beyond the Hessian level can be found in [10, 23, 24]. A detailed analysis of the stability of critical points of V_{BH} in (the large volume limit of) compactifications of Type IIA superstrings on CY_3s has been recently given in [34].

In the present review, we will consider only critical points of V_{BH} ($\frac{1}{2}$-BPS as well as non-BPS) which are *non-degenerate*, i.e. with a finite, non-vanishing horizon area, corresponding to the so-called "*large*" BHs[4].

Due to the so-called Attractor Mechanism (AM) [1, 2, 3, 4, 5], the Bekenstein-Hawking entropy [69, 70, 71, 72, 73] of "*large*" extremal BHs can be obtained by extremizing $V_{BH}(\phi, Q)$, where "ϕ" now denotes the set of real scalars relevant for the AM, and Q is defined by (1). In $\mathscr{N}=2,\ d=4$ supergravity, non-degenerate attractor horizon geometries correspond to BH solitonic states belonging to $\frac{1}{2}$-BPS "short massive multiplets" or to non-BPS "long massive multiplets", respectively. The BPS bound [74] requires that

$$M_{ADM} \geq |Z|, \tag{13}$$

where M_{ADM} denotes the Arnowitt-Deser-Misner (ADM) mass [75]. At the event horizon, extremal BPS BHs do saturate such a bound, whereas the non-BPS ones satisfy[5]

$$\begin{aligned} &\frac{1}{2}\text{-}BPS\colon 0 < |Z|_H = M_{ADM,H};\\ &non\text{-}BPS \begin{cases} Z \neq 0\colon 0 < |Z|_H < M_{ADM,H};\\ Z = 0\colon 0 = |Z|_H < M_{ADM,H}, \end{cases} \end{aligned} \tag{14}$$

where $M_{ADM,H}$ is obtained by extremizing $V_{BH}(\phi, Q)$ with respect to its dependence on the scalars:

$$M_{ADM,H}(Q) = \sqrt{V_{BH}(\phi, Q)|_{\partial_\phi V_{BH}=0}}. \tag{15}$$

The (purely) charge-dependent BH entropy S_{BH} is given by the Bekenstein-Hawking entropy-area formula [5, 69, 70, 71, 72, 73]

$$S_{BH}(Q) = \frac{A_H(Q)}{4} = \pi\, V_{BH}(\phi, Q)|_{\partial_\phi V_{BH}=0} = \pi V_{BH}(\phi_H(Q), Q), \tag{16}$$

where A_H is the area of the BH event horizon.

Non-degenerate, non-supersymmetric (non-BPS) extremal BH (and black string) attractors arise also in $\mathscr{N}=2,\ d=5,\ 6$ supergravity and in $\mathscr{N}>2,\ d=4,\ 5,\ 6$ extended supergravities (see e.g. [19, 38, 40, 46, 49, 76, 77, 78, 79, 80], and

The issue of stability of non-BPS critical points of V_{BH} in homogeneous (not necessarily symmetric) $\mathscr{N}=2,\ d=4$ special Kähler geometries has been treated exhaustively in [36]. It was derived that all non-BPS critical points of V_{BH} in such geometries are stable, up to a certain number of "flat" directions (present at all order in the covariant differentiation of V_{BH}), which span a certain moduli space, pertaining to the considered class of solutions of the attractor equations. The results of [36] hold in general for any theory (not necessarily involving supersymmetry) in which gravity is coupled to Abelian gauge vectors and with a scalar sigma model endowed with homogeneous geometry (see further below in the present lectures).

[4] For further elucidations, we refer the reader, e.g. to the recent lectures of Sen [47], where important aspects of BH attractors are presented: the microscopic string theory counting of states explaining the macroscopic BH entropy and the treatment of higher-derivative terms in the actions.

[5] Here and in what follows, the subscript "H" will denote values at the BH event horizon.

Refs. therein). In the present lectures, we will focus on extremal BH attractors in $\mathscr{N}=2,\ d=4$ ungauged supergravity coupled to Abelian vector multiplets, where the scalar manifold parameterized by the scalars is endowed with the so-called special Kähler (SK) geometry (see Sect. 2).

Flux vacua (FV) became recently one of the new playgrounds for string theory, in general, and in particular, in the context of moduli stabilization (for an introduction to flux compactifications, see e.g. [81, 82, 83, 84, 85, 86] and Refs. therein).

The advances of observational cosmology and the emergence of the so-called "*standard cosmological model*" enforce on string theory/supergravity a responsibility to address the current and future observations. This requires a solution of the problem of moduli stabilization. In the early Universe, during inflation, all string theory moduli but the inflaton have to be stabilized, in order to produce an effective four-dimensional General Relativity and also in order for inflation to explain the cosmic microwave background observations. At the present time, all moduli have to be stabilized in a four-dimensional de Sitter space to explain dark energy and acceleration of the Universe which took place during the last few billion years.

The procedure of moduli stabilization in string theory consists of few steps.

One of the steps is the stabilization of moduli by fluxes in type IIB string theory, determining $d=4$FV, with effective $\mathscr{N}=1$ local supersymmetry and complex structure moduli stabilized; an important feature of such a procedure is the non-stabilization of the Kähler moduli. However, the largest contribution to the counting of the Calabi-Yau vacua in the so-called *String Landscape* comes from the diversity of FV.

Thus, it is still interesting to study the mechanism of stabilization of the axion-dilaton and complex structure moduli in FV, ignoring the Kähler moduli. We will deal with such a scenario, in the particular case in which the geometry of the complex structure moduli is SK and not simply Kähler. In such a framework, it turns out that the equations determining the FV configurations are closely related to the abovementioned extremal BH attractor equations of $\mathscr{N}=2,\ d=4$ supergravity.

In the studies of FV one can start with an F-theory flux compactification on an elliptically fibered Calabi-Yau fourfold CY_4 in the orientifold limit in which $CY_4=\frac{CY_3\times T^2}{\mathbb{Z}_2}$, where T^2 is the two-torus. In type IIB string theory, this is equivalent to compactifying on the orientifold limit of CY_3. The resulting low energy, $d=4$ effective theory is $\mathscr{N}=1$ supergravity, where the information on string theory choice of compactification is encoded into a flux superpotential W and a Kähler potential K. As explained above, we assume that both flux superpotential W and Kähler potential K depend only on the complex structure (CS) moduli of CY_4, spanning the CS moduli space M. Because of the orientifold limit of CY_4, M has the product structure $\mathscr{M}=\mathscr{M}_{CS}(CY_3)\times\mathscr{M}_{\tau}$ (see (200) further below), where $\mathscr{M}_{CS}(CY_3)$ (simply named $\mathscr{M}_{CS}$ further below) is the CS moduli space of CY_3 and $\mathscr{M}_{\tau}$ is the moduli space of the elliptic curve T^2 spanned by the axion-dilaton τ (named t^0 in the treatment of the present lectures). Let us just mention here that in order to stabilize also the Kähler moduli of CY_4, one should incorporate the non-perturbative string effects (see e.g. [87]), which however we will not discuss here.

The potential in the effective $\mathscr{N}=1,\ d=4$ supergravity theory, in the Planckian units set equal to one, is given by [88, 89]

$$V_{\mathscr{N}=1}=e^{K}\left(\sum_{A=0}^{h_{2,1}(CY_3)}|D_AW|^2-3|W|^2\right)=\sum_{A=0}^{h_{2,1}(CY_3)}|D_AZ|^2-3|Z|^2, \tag{17}$$

where $A=0$ refers to the axion-dilaton $\tau\equiv t^0$ and $A=i\in\{1,\cdots,h_{2,1}(CY_3)\}$ to the CS moduli t^i of CY_3 $\left(h_{2,1}\equiv dim\left(H^{2,1}\left(CY_3\right)\right)\right.$; see Sect. 4.1). We defined $Z\equiv e^{\frac{K}{2}}W$, for the extremal BH attractors in $\mathscr{N}=2,\ d=4$ supergravity, even if the analogy is only formal, because in the present $d=4$ framework with $\mathscr{N}=1$ local supersymmetry there is no central charge at all.

The real Kähler potential of the effective $\mathscr{N}=1,\ d=4$ supergravity theory reads (see (231) below)

$$K=-ln\langle\Omega_4,\bar{\Omega}_4\rangle=-ln\left(\langle\Omega_1,\bar{\Omega}_1\rangle\right)-ln\left(\langle\Omega_3,\bar{\Omega}_3\rangle\right), \tag{18}$$

where Ω_4 is a nowhere vanishing holomorphic 4-form defined on CY_4. In the orientifold limit, Ω_4 is a product of an appropriate holomorphic 3-form Ω_3 of CY_3 and the holomorphic 1-form Ω_1 of the torus T^2 (see Sect. 4.1.2).

The flux holomorphic superpotential W is defined as a section of the line bundle $\mathscr{L}$ by [90, 91, 92] (see (232) below)

$$W\equiv Ze^{-\frac{K}{2}}=\langle\mathscr{F}_4,\Omega_4\rangle\equiv\int_{CY_4}\mathscr{F}_4\wedge\Omega_4, \tag{19}$$

where $\mathscr{F}_4\in H^4(CY_4)$ is the 4-form flux.

In generic local "flat" coordinates of M (with 0 and A-indices, respectively, referring to the axion-dilaton and CS moduli of CY_3), $\mathscr{F}_4$ enjoys the following Hodge decomposition, which we present here in terms of Z for the sake of comparison with its "BH-counterpart" (i.e. the Hodge decomposition (5) of the 3-form flux $\mathscr{H}_3$) ($\hat{\Omega}_4\equiv e^{\frac{1}{2}K}\Omega_4$; see definition (234) and (244) below):

$$\mathfrak{F}_4=2Re\left[\overline{Z}\hat{\Omega}_4-\delta^{A\overline{B}}\left(\overline{D}_{\overline{B}}\overline{Z}\right)D_A\hat{\Omega}_4+\delta^{A\overline{B}}\left(\overline{D}_{\overline{\underline{0}}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right]. \tag{20}$$

By imposing the supersymmetry-preserving condition $DZ=0$ (formally identical to the one appearing in the abovementioned theory of extremal BH attractors in $\mathscr{N}=2,\ d=4$ supergravity), the identity (20) becomes a supersymmetric FV Attractor equation. Indeed, the left-hand side (l.h.s.) depends on fluxes and the r.h.s. depends on axion-dilaton and on CS moduli of CY_3; thus, the solution stabilizes the axion-dilaton and the CS moduli of CY_3 purely in terms of fluxes: beside (7), one gets

$$\mathfrak{F}_4=2Re\left[\overline{Z}\hat{\Omega}_4+\delta^{A\overline{B}}\left(\overline{D}_{\overline{\underline{0}}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right]_{DZ=0}. \tag{21}$$

Let us now compare the supersymmetric FV Attractor Equations (21) with their "BH-counterpart", i.e. with the BPS extremal BH Attractor Equations (7).

In the case of FV, instead of the imaginary part we have a real part of a somewhat analogous expression: this is due to the fact that for FV one has a 4-form flux $\mathfrak{F}_4$ on a CY_3 orientifold rather than of a 3-form flux $\mathscr{H}_3$ on CY_3.

The other significant difference is in the second term in the r.h.s. of (21). This term, absent in the BH case, is proportional to the second-order covariant derivative of Z along the τ direction and one of the directions pertaining to the CS moduli of CY_3. In the BH case, there is a relation $D_iD_jZ = iC_{ijk}\overline{D}^k\overline{Z}$ (see the second of (38) below), and therefore the second covariant derivative of Z is not an independent term for BH, differently from $D_{\underline{0}}D_IZ$ in the FV case, which is an independent term in the decomposition of forms.

The absence of such a term in the BPS extremal BH Attractor Equations (7) does not allow for non-degenerate BPS extremal BH attractors with vanishing central charge: Indeed, on the BH side $Z = 0$ and $DZ = 0$ yield $V_{BH} = 0$. This limit case corresponds to a classical "*small*" extremal BH, exhibiting a naked singularity because the area of the BH event horizon vanishes. The Attractor Mechanism in such a case simply ceases to hold, because for $Z = 0$ and $DZ = 0$ the BPS extremal BH Attractor Equations (3) admit as unique solution $Q = 0$.

The same does not happen on the FV side. Indeed, by substituting $Z = 0$ and $DZ = 0$ into the Hodge decomposition (20) does not generate any unconsistency: The second term in the r.h.s. of (21) provides a consistent solution for superymmetric Minkowski vacua (with $DZ = 0$ and $V_{\mathscr{N}=1} = 0$). Of course, more general supersymmetric solutions with $Z \neq 0$ are allowed, and they correspond to supersymmetric AdS FV (see e.g. [11, 93]).

The aim of the present paper is to show the Attractor Mechanism at work in two completely different $d = 4$ frameworks: extremal BH in $\mathscr{N} = 2$ supergravity and $\mathscr{N} = 1$ flux compactifications.

The plan of the paper is as follows.

In Sect. 2 we recall the fundamentals of the special Kähler geometry, underlying the vector multiplets' scalar manifold of $\mathscr{N} = 2,\ d = 4$ ungauged supergravity, as well as the complex structure moduli space of certain $\mathscr{N} = 1,\ d = 4$ supergravities obtained by consistently orientifolding of $\mathscr{N} = 2$ theories, such as Type IIB compactified on $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$.

Section 3 gives an introduction to the issue of the Attractor Mechanism in the framework where it was originally discovered by Ferrara, Kallosh and Strominger [1, 2, 3, 4, 5], namely in the stabilization of the vector multiplet' scalars near the event horizon of an extremal, static, spherically symmetric and asymptotically flat BH in $\mathscr{N} = 2,\ d = 4$ ungauged supergravity.

Section 3.1 presents the so-called *"criticality conditions"* approach to the Attractor Mechanism, in which the purely charge-dependent stabilized configurations of the scalars at the BH horizon can be computed as the critical points of a certain real positive BH effective potential V_{BH}, whose classification is given in Sect. 3.1.1. The stability of the critical points of V_{BH} is then analyzed in Sect. 3.2, both in the general case of n_V moduli (Sect. 3.2.1) and in the 1-modulus case (Sect. 3.2.2).

Section 3.3 presents another, equivalent approach to the Attractor Mechanism, recently named *"New Attractor"* approach. In Sect. 3.2.2 it is exploited in a general

$\mathcal{N}=2,\ d=4$ supergravity framework, by substituting the (various classes of) criticality conditions of V_{BH} into some geometrical identities of special Kähler geometry, expressing nothing but a change of basis between *"dressed"* and *"undressed"* charges and derived in Sect. 3.3.1.

Section 3.4 implements the *"New Attractor"* approach in a stringy framework, namely in Type IIB compactified on CY_3. In Sect. 3.4.2 the (various classes of) criticality conditions of V_{BH} are inserted into some general identities (equivalent to the identities derived in Sect. 3.3.1), expressing the decomposition of the real, Kähler gauge-invariant 3-form flux $\mathscr{H}_3$ along the third Dalbeault cohomogy of CY_3 and derived in Sect. 3.4.1.

Section 4 deals with the Attractor Mechanism in a completely different framework, namely in $\mathcal{N}=1,\ d=4$ ungauged supergravity obtained by consistently orientifolding the $\mathcal{N}=2$ theory and thus mantaining a special Kähler geometry of the manifold of the scalars surviving the orientifolding. In the considered example of Type IIB associated with $\frac{CY_3\times T^2}{\mathbb{Z}_2}$, the Attractor Mechanism determines the stabilization of the universal axion-dilaton and of the complex structure moduli in terms of the Ramond-Ramond (RR) and Neveu-Schwarz-Neveu-Schwarz (NSNS) fluxes.

Section 4.1 introduces the fundamentals of the geometry of (the moduli space of) CY_3 orientifolds: the vielbein and the metric tensor (Sect. 4.1.1), the relevant 1-, 3- and 4-forms (Sect. 4.1.2), and the Hodge decomposition of the real, Kähler gauge-invariant 4-form flux $\mathfrak{F}_4$, unifying the RR 3-form flux $\mathfrak{F}_3$ with the NSNS 3-form flux $\mathfrak{H}_3$ (Sect. 4.1.3).

Section 4.2 presents the so-called *"criticality conditions"* approach to the Attractor Mechanism in flux vacua (FV) compactifications of the kind considered above, in which the (complex structure) moduli space is endowed with special Kähler geometry. The purely flux-dependent stabilized vacuum configurations of the axion-dilaton and complex structure moduli can be computed as the critical points of a certain real (not necessarily positive) FV effective potential $V_{\mathcal{N}=1}$. Since (differently from its BH $\mathcal{N}=2$ counterpart V_{BH}) $V_{\mathcal{N}=1}$ has no definite sign, the FV attractor configurations can correspond to a de Sitter (dS, $V_{\mathcal{N}=1}>0$), Minkowski ($V_{\mathcal{N}=1}=0$) or anti-de Sitter (AdS, $V_{\mathcal{N}=1}<0$) vacuum.

Finally, Sect. 4.3 implements the *"New Attractor"* approach to the Attractor mechanism in the considered class of FV compactifications, in the case of supersymmetric vacuum configurations. The supersymmetric criticality conditions of $V_{\mathcal{N}=1}$ are inserted into the Hodge decomposition of the 4-form flux $\mathfrak{F}_4$, and the resulting supersymmetric FV Attractor Equations lead to the classification of the supersymmetric FV into three general families.

Two Appendices, containing technical details, conclude the lectures.

2 Special Kähler Geometry

In the present section, we briefly recall the fundamentals of the SK geometry underlying the scalar manifold $\mathscr{M}_{n_V}$ of $\mathcal{N}=2,\ d=4$ supergravity, n_V being the number of Abelian vector supermultiplets coupled to the supergravity multiplet $(dim_{\mathbb{C}}\mathscr{M}_{n_V} = n_V)$ (see e.g. [63, 64, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104]).

It is convenient to switch from the Riemannian $2n_V$-dim. parameterization of $\mathscr{M}_{n_V}$ given by the local real coordinates $\{\phi^a\}_{a=1,\dots,2n_V}$ to the Kähler n_V-dim. holomorphic/antiholomorphic parameterization given by the local complex coordinates $\left\{z^i, \overline{z}^{\overline{i}}\right\}_{i,\overline{i}=1,\dots,n_V}$. This corresponds to the following *unitary Cayley transformation*:

$$z^k \equiv \frac{\varphi^{2k-1} + i\varphi^{2k}}{\sqrt{2}}, \quad k = 1, \dots, n_V. \tag{22}$$

The metric structure of $\mathscr{M}_{n_V}$ is given by the covariant (special) Kähler metric tensor $g_{i\overline{j}}(z,\overline{z}) = \partial_i \overline{\partial}_{\overline{j}} K(z,\overline{z})$, $K(z,\overline{z})$ being the real Kähler potential.

Usually, the $n_V \times n_V$ Hermitian matrix $g_{i\overline{j}}$ is assumed to be non-degenerate (i.e. invertible, with non-vanishing determinant and rank n_V) and with strict positive Euclidean signature (i.e. with all strictly positive eigenvalues) *globally* in $\mathscr{M}_{n_V}$. We will so assume, even though we will be concerned mainly with the properties of $g_{i\overline{j}}$ at those peculiar points of $\mathscr{M}_{n_V}$ which are critical points of V_{BH}.

It is worth remarking here that various possibilities arise when going beyond the assumption of *global strict regular* $g_{i\overline{j}}$, namely:

– (*locally*) *not strictly regular* $g_{i\overline{j}}$, i.e. a (*locally*) non-invertible metric tensor, with some strictly positive and some vanishing eigenvalues (rank $< n_V$);
– (*locally*) *non-regular non-degenerate* $g_{i\overline{j}}$, i.e. a (*locally*) invertible metric tensor with *pseudo-Euclidean signature*, namely with some strictly positive and some strictly negative eigenvalues (rank $= n_V$);
– (*locally*) *non-regular degenerate* $g_{i\overline{j}}$, i.e. a (*locally*) non-invertible metric tensor with some strictly positive, some strictly negative, and some vanishing eigenvalues (rank $< n_V$).

The *local* violation of strict regularity of $g_{i\overline{j}}$ would produce some kind of "phase transition" in the SKG endowing $\mathscr{M}_{n_V}$, corresponding to a breakdown of the 1-dim. effective Lagrangian picture (see [5, 105] and also [18] and [80]) of $d = 4$ (extremal) BHs obtained by integrating all massive states of the theory out, unless new massless states appear [5].

The previously mentioned $\mathscr{N} = 2$, $d = 4$ covariantly holomorphic *central charge function* is defined as

$$\begin{aligned} Z(z,\overline{z};q,p) &\equiv Q\epsilon V(z,\overline{z}) = q_\Lambda L^\Lambda(z,\overline{z}) - p^\Lambda M_\Lambda(z,\overline{z}) = e^{\frac{1}{2}K(z,\overline{z})} Q\epsilon \Pi(z) \\ &= e^{\frac{1}{2}K(z,\overline{z})}\left[q_\Lambda X^\Lambda(z) - p^\Lambda F_\Lambda(z)\right] \equiv e^{\frac{1}{2}K(z,\overline{z})} W(z;q,p), \end{aligned} \tag{23}$$

where ϵ is the $(2n_V + 2)$-dim. square symplectic metric (subscripts denote dimensions of square sub-blocks)

$$\epsilon \equiv \begin{pmatrix} 0_{n_V+1} & -\mathbb{I}_{n_V+1} \\ \\ \mathbb{I}_{n_V+1} & 0_{n_V+1} \end{pmatrix}, \tag{24}$$

and $V(z,\bar{z})$ and $\Pi(z)$, respectively, stand for the $(2n_V+2)\times 1$ covariantly holomorphic (Kähler weights $(1,-1)$) and holomorphic (Kähler weights $(2,0)$) period vectors in symplectic basis:

$$\overline{D}_{\bar{i}}V(z,\bar{z}) = \left(\overline{\partial}_{\bar{i}} - \frac{1}{2}\overline{\partial}_{\bar{i}}K\right)V(z,\bar{z}) = 0, \quad D_iV(z,\bar{z}) = \left(\partial_i + \frac{1}{2}\partial_iK\right)V(z,\bar{z})$$

$$\Updownarrow$$

$$V(z,\bar{z}) = e^{\frac{1}{2}K(z,\bar{z})}\Pi(z), \ \overline{D}_{\bar{i}}\Pi(z) = \overline{\partial}_{\bar{i}}\Pi(z) = 0, \quad D_i\Pi(z) = (\partial_i + \partial_iK)\Pi(z),$$

$$\Pi(z) \equiv \begin{pmatrix} X^{\Lambda}(z) \\ F_{\Lambda}(X(z)) \end{pmatrix} = exp\left(-\frac{1}{2}K(z,\bar{z})\right)\begin{pmatrix} L^{\Lambda}(z,\bar{z}) \\ M_{\Lambda}(z,\bar{z}) \end{pmatrix}, \tag{25}$$

with $X^{\Lambda}(z)$ and $F_{\Lambda}(X(z))$ being the holomorphic sections of the $U(1)$ line (Hodge) bundle over $\mathscr{M}_{n_V}$. $W(z;q,p)$ is the so-called *holomorphic* $\mathscr{N}=2$, $d=4$ *central charge function*, also named $\mathscr{N}=2$ *superpotential*.

Up to some particular choices of local symplectic coordinates in $\mathscr{M}_{n_V}$, the covariant symplectic holomorphic sections $F_{\Lambda}(X(z))$ may be seen as derivatives of an *holomorphic prepotential* function F (with Kähler weights $(4,0)$):

$$F_{\Lambda}(X(z)) = \frac{\partial F(X(z))}{\partial X^{\Lambda}}. \tag{26}$$

In $\mathscr{N}=2$, $d=4$ supergravity the holomorphic function F is constrained to be homogeneous of degree 2 in the contravariant symplectic holomorphic sections $X^{\Lambda}\ (z)$, i.e. (see [64] and Refs. therein)

$$2F(X(z)) = X^{\Lambda}(z)F_{\Lambda}(X(z)). \tag{27}$$

The normalization of the holomorphic period vector $\Pi(z)$ is such that

$$\begin{aligned} K(z,\bar{z}) &= -ln\left[i\left\langle \Pi(z), \overline{\Pi}(\bar{z})\right\rangle\right] \equiv -ln\left[i\Pi^T(z)\epsilon\overline{\Pi}(\bar{z})\right] \\ &= -ln\left\{i\left[\overline{X}^{\Lambda}(\bar{z})F_{\Lambda}(z) - X^{\Lambda}(z)\overline{F}_{\Lambda}(\bar{z})\right]\right\}, \end{aligned} \tag{28}$$

where $\langle\cdot,\cdot\rangle$ stands for the symplectic scalar product defined by ϵ.

Note that under a Kähler transformation

$$K(z,\bar{z}) \longrightarrow K(z,\bar{z}) + f(z) + \overline{f}(\bar{z}) \tag{29}$$

($f(z)$ being a generic holomorphic function), the holomorphic period vector transforms as

$$\Pi(z) \longrightarrow \Pi(z)\, e^{-f(z)} \Leftrightarrow X^{\Lambda}(z) \longrightarrow X^{\Lambda}(z)\, e^{-f(z)}. \tag{30}$$

This means that, at least locally, the contravariant holomorphic symplectic sections $X^{\Lambda}(z)$ can be regarded as a set of homogeneous coordinates on $\mathscr{M}_{n_V}$, provided the Jacobian complex $n_V \times n_V$ holomorphic matrix

$$e_i^a(z) \equiv \frac{\partial}{\partial z^i}\left(\frac{X^a(z)}{X^0(z)}\right), \quad a = 1, \ldots, n_V \tag{31}$$

is invertible. If this is the case, then one can introduce the local projective symplectic coordinates

$$t^a(z) \equiv \frac{X^a(z)}{X^0(z)}, \tag{32}$$

and the SKG of $\mathscr{M}_{n_V}$ turns out to be based on the holomorphic prepotential $\mathscr{F}(t) \equiv \left(X^0\right)^{-2} F(X)$. By using the t-coordinates, (28) can be rewritten as follows ($\mathscr{F}_a(t) = \partial_a \mathscr{F}(t)$, $\bar{t}^a = \overline{t^a}$, $\overline{\mathscr{F}}_a(\bar{t}) = \overline{\mathscr{F}_a(t)}$):

$$K(t,\bar{t}) = -ln\left\{i\left|X^0(z(t))\right|^2\left[2\left(\mathscr{F}(t) - \overline{\mathscr{F}}(\bar{t})\right) - \left(t^a - \bar{t}^a\right)\left(\mathscr{F}_a(t) + \overline{\mathscr{F}}_a(\bar{t})\right)\right]\right\}. \tag{33}$$

By performing a Kähler gauge-fixing with $f(z) = ln\left(X^0(z)\right)$, yielding that $X^0(z) \longrightarrow 1$, one thus gets

$$K(t,\bar{t})|_{X^0(z)\longrightarrow 1} = -ln\left\{i\left[2\left(\mathscr{F}(t) - \overline{\mathscr{F}}(\bar{t})\right) - \left(t^a - \bar{t}^a\right)\left(\mathscr{F}_a(t) + \overline{\mathscr{F}}_a(\bar{t})\right)\right]\right\}. \tag{34}$$

In particular, one can choose the so-called *special coordinates*, i.e. the system of local projective t-coordinates such that

$$e_i^a(z) = \delta_i^a \Leftrightarrow t^a(z) = z^i\left(+c^i, c^i \in \mathbb{C}\right). \tag{35}$$

Thus, (34) acquires the form

$$\begin{aligned} K(t,\bar{t})|_{X^0(z)\longrightarrow 1, e_i^a(z)=\delta_i^a} = &-ln\left\{i\left[2\left(\mathscr{F}(z) - \overline{\mathscr{F}}(\bar{z})\right) - \left(z^j - \bar{z}^{\bar{j}}\right)\right.\right. \\ &\times \left.\left.\left(\mathscr{F}_j(z) + \overline{\mathscr{F}}_{\bar{j}}(\bar{z})\right)\right]\right\}. \end{aligned} \tag{36}$$

Moreover, it should be recalled that Z has Kähler weights $(p,\overline{p}) = (1,-1)$, and therefore its Kähler-covariant derivatives read

$$D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K\right) Z,\ \overline{D}_{\bar{i}} Z = \left(\overline{\partial}_{\bar{i}} - \frac{1}{2}\overline{\partial}_{\bar{i}} K\right) Z. \tag{37}$$

The fundamental differential relations of SK geometry are[6] (see e.g. [64]):

$$\begin{cases} D_i Z = Z_i; \\ D_i Z_j = iC_{ijk} g^{k\bar{k}} \overline{D}_{\bar{k}} \overline{Z} = iC_{ijk} g^{k\bar{k}} \overline{Z}_{\bar{k}}; \\ D_i \overline{D}_{\bar{j}} \overline{Z} = D_i \overline{Z}_{\bar{j}} = g_{i\bar{j}} \overline{Z}; \\ D_i \overline{Z} = 0, \end{cases} \tag{38}$$

where the first relation is nothing but the definition of the so-called *matter charges* Z_i, and the fourth relation expresses the Kähler-covariant holomorphicity of Z. C_{ijk} is the rank-3, completely symmetric, covariantly holomorphic tensor of SK geometry (with Kähler weights $(2,-2)$) (see e.g.[7] [64, 99, 100, 101, 102, 103]):

$$\begin{cases} C_{ijk} = \left\langle D_i D_j V, D_k V \right\rangle = e^K \left(\partial_i \mathcal{N}_{\Lambda\Sigma}\right) D_j X^\Lambda D_k X^\Sigma \\ = e^K \left(\partial_i X^\Lambda\right) \left(\partial_j X^\Sigma\right) \left(\partial_k X^\Xi\right) \partial_\Xi \partial_\Sigma F_\Lambda (X) \equiv e^K W_{ijk},\ \overline{\partial}_{\bar{l}} W_{ijk} = 0; \\ C_{ijk} = D_i D_j D_k \mathcal{S},\ \mathcal{S} \equiv -iL^\Lambda L^\Sigma Im\left(F_{\Lambda\Sigma}\right),\ F_{\Lambda\Sigma} \equiv \dfrac{\partial F_\Lambda}{\partial X^\Sigma}, F_{\Lambda\Sigma} \equiv F_{(\Lambda\Sigma)}; \\ C_{ijk} = -i g_{i\bar{l}} \overline{f}^{\bar{l}}_{\Lambda} D_j D_k L^\Lambda, \quad \overline{f}^{\bar{l}}_{\Lambda} \left(\overline{DL}^{\Lambda}_{\bar{s}}\right) \equiv \delta^{\bar{l}}_{\bar{s}}; \end{cases} \tag{39}$$

$\overline{D}_{\bar{i}} C_{jkl} = 0$ (*covariant holomorphicity*);

$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}} g_{k\bar{l}} - g_{i\bar{l}} g_{k\bar{j}} + C_{ikp} \overline{C}_{\bar{j}\bar{l}\bar{p}} g^{p\bar{p}}$(usually named *SKG constraints*) ;

$D_{[i} C_{j]kl} = 0,$

where the last property is a consequence, through the SKG constraints and the covariant holomorphicity of C_{ijk}, of the Bianchi identities for the Riemann tensor $R_{i\bar{j}k\bar{l}}$ (see e.g. [99, 100]), and square brackets denote antisymmetrization with respect to enclosed indices. For later convenience, here it is worth writing the expression for the holomorphic covariant derivative of C_{ijk}:

$$D_i C_{jkl} = D_{(i} C_{j)kl} = \partial_i C_{jkl} + \left(\partial_i K\right) C_{jkl} + \Gamma^{\ m}_{ij} C_{mkl} + \Gamma^{\ m}_{ik} C_{mjl} + \Gamma^{\ m}_{il} C_{mjk}. \tag{40}$$

It is worth recalling that in a generic Kähler geometry $R_{i\bar{j}k\bar{l}}$ reads

---

[6] Actually, there are different (equivalent) defining approaches to SK geometry. For subtleties and further elucidation concerning such an issue, see e.g. [106] and [107].

[7] Notice that the third of (39) correctly defines the Riemann tensor $R_{i\bar{j}k\bar{l}}$, and it is actually the opposite of the one which may be found in a large part of existing literature. Such a formulation of the so-called *SKG constraints* is well defined, because, as we mention at the end of Sect. 3.2, it yields negative values of the constant scalar curvature of (n_V = 1-dim.) homogeneous symmetric compact SK manifolds.

$$
\begin{aligned}
&R_{i\bar{j}k\bar{l}} = g^{m\bar{n}}\left(\overline{\partial}_{\bar{l}}\overline{\partial}_{\bar{j}}\partial_m K\right)\partial_i\overline{\partial}_{\bar{n}}\partial_k K - \overline{\partial}_{\bar{l}}\partial_i\overline{\partial}_{\bar{j}}\partial_k K = g_{k\bar{n}}\partial_i\overline{\Gamma}^{\bar{n}}_{\bar{l}\bar{j}} = g_{n\bar{l}}\overline{\partial}_{\bar{j}}\Gamma^{\,n}_{ki},\\
&\overline{R_{i\bar{j}k\bar{l}}} = R_{\bar{j}i\bar{l}k} \quad (\textit{reality}),\\
&\Gamma^{\,l}_{ij} = -g^{l\bar{l}}\partial_i g_{j\bar{l}} = -g^{l\bar{l}}\partial_i\overline{\partial}_{\bar{l}}\partial_j K = \Gamma^{\,l}_{(ij)},
\end{aligned} \tag{41}
$$

where ${\Gamma_{ij}}^{l}$ stand for the Christoffel symbols of the second kind of the Kähler metric $g_{i\bar{j}}$.

In the first of (39), a fundamental entity, the so-called kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ of $\mathcal{N}=2,\ d=4$ supergravity, has been introduced (see also (168) further below). It is an $(n_V+1)\times(n_V+1)$ complex symmetric, moduli-dependent, Kähler gauge-invariant matrix defined by the following fundamental *Ansätze* of SKG, solving the *SKG constraints* (given by the third of (39)):

$$
M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma,\ D_i M_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}D_i L^\Sigma. \tag{42}
$$

By introducing the $(n_V+1)\times(n_V+1)$ complex matrices $(I=1,\dots,n_V+1)$

$$
f_I^\Lambda(z,\bar{z}) \equiv \left(\overline{D}_{\bar{i}}\overline{L}^\Lambda(z,\bar{z}), L^\Lambda(z,\bar{z})\right),\ h_{I\Lambda}(z,\bar{z}) \equiv \left(\overline{D}_{\bar{i}}\overline{M}_\Lambda(z,\bar{z}), M_\Lambda(z,\bar{z})\right), \tag{43}
$$

the *Ansätze* (42) univoquely determine $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ as

$$
\mathcal{N}_{\Lambda\Sigma}(z,\bar{z}) = h_{I\Lambda}(z,\bar{z})\circ\left(f^{-1}\right)^I_\Sigma(z,\bar{z}), \tag{44}
$$

where $\circ$ denotes the usual matrix product, and $\left(f^{-1}\right)^I_\Sigma f_I^\Lambda = \delta^\Lambda_\Sigma,\ \left(f^{-1}\right)^I_\Lambda f_J^\Lambda = \delta^I_J$.

The covariantly holomorphic $(2n_V+2)\times 1$ period vector $V(z,\bar{z})$ is *symplectically orthogonal* to all its Kähler-covariant derivatives:

$$
\begin{cases}
\langle V(z,\bar{z}), D_i V(z,\bar{z})\rangle = 0;\\
\left\langle V(z,\bar{z}), \overline{D}_{\bar{i}} V(z,\bar{z})\right\rangle = 0;\\
\left\langle V(z,\bar{z}), D_i \overline{V}(z,\bar{z})\right\rangle = 0;\\
\left\langle V(z,\bar{z}), \overline{D}_{\bar{i}} \overline{V}(z,\bar{z})\right\rangle = 0.
\end{cases} \tag{45}
$$

Morover, it holds that

$$
\begin{aligned}
g_{i\bar{j}}(z,\bar{z}) &= -i\left\langle D_i V(z,\bar{z}), \overline{D}_{\bar{j}}\overline{V}(z,\bar{z})\right\rangle\\
&= -2Im\left(\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})\right)D_i L^\Lambda(z,\bar{z})\overline{D}_{\bar{i}}\overline{L}^\Sigma(z,\bar{z}) = 2Im\left(F_{\Lambda\Sigma}(z)\right)D_i L^\Lambda(z,\bar{z})\overline{D}_{\bar{i}}\overline{L}^\Sigma(z,\bar{z});
\end{aligned} \tag{46}
$$

$$
\left\langle V(z,\bar{z}), D_i\overline{D}_{\bar{j}} V(z,\bar{z})\right\rangle = iC_{ijk}g^{k\bar{k}}\left\langle V(z,\bar{z}), \overline{D}_{\bar{k}}\overline{V}(z,\bar{z})\right\rangle = 0. \tag{47}
$$

3 Extremal Black Hole Attractor Equations in $\mathscr{N}=2$, $d=4$ (ungauged) Supergravity

3.1 Black Hole Effective Potential and "Criticality Conditions" Approach

In $\mathscr{N}=2,\ d=4$ supergravity the "effective BH potential" reads [3, 4, 64]

$$\begin{aligned}V_{BH}\left(z,\bar{z};q,p\right)&=\left|Z\right|^{2}\left(z,\bar{z};q,p\right)+g^{i\bar{j}}\left(z,\bar{z}\right)D_{i}Z\left(z,\bar{z};q,p\right)\overline{D}_{\bar{j}}\overline{Z}\left(z,\bar{z};q,p\right)\\&=I_{1}\left(z,\bar{z};q,p\right)\geqslant 0,\end{aligned}\tag{48}$$

where I_1 is the first, positive-definite real invariant I_1 of SK geometry (see e.g. [23, 64]). It should be noticed that V_{BH} can also be obtained by left-multiplying the SKG vector identity (146) by the $1\times(2n_V+2)$ complex moduli-dependent vector $-\frac{1}{2}Q\mathscr{M}(\mathscr{N})$; indeed, since the matrix $\mathscr{M}(\mathscr{N})$ is symplectic, one finally gets [3, 4, 64]

$$V_{BH}\left(z,\bar{z};q,p\right)=-\frac{1}{2}Q\mathscr{M}\left(\mathscr{N}\right)Q^{T}.\tag{49}$$

It is interesting to remark that the result (49) can be elegantly obtained from the SK geometry identities (146) by making use of the following relation (see [19], where a generalization for $\mathscr{N}>2$-extended supergravities is also given):

$$\frac{1}{2}\left(\mathscr{M}\left(\mathscr{N}\right)+i\Omega\right)\mathscr{V}=i\Omega\mathscr{V}\Leftrightarrow\mathscr{M}\left(\mathscr{N}\right)\mathscr{V}=i\Omega\mathscr{V},\tag{50}$$

where $\mathscr{V}$ is a $(2n_V+2)\times(n_V+1)$ matrix defined as follows:

$$\mathscr{V}\equiv\left(V,\overline{D}_{\bar{1}}\overline{V},\ldots,\overline{D}_{\overline{n_V}}\overline{V}\right).\tag{51}$$

By differentiating (48) with respect to the moduli, the criticality conditions of V_{BH} can be easily shown to acquire the form [5]

$$D_{i}V_{BH}=\partial_{i}V_{BH}=0\Leftrightarrow 2\overline{Z}D_{i}Z+g^{j\bar{j}}\left(D_{i}D_{j}Z\right)\overline{D}_{\bar{j}}\overline{Z}=0.\tag{52}$$

These are the what one should rigorously refer to as the $\mathscr{N}=2,\ d=4$ supergravity Attractor Equations (AEs).

In the present work, we will call *AEs* also some *geometrical identities* evaluated along the criticality conditions of the relevant "effective potential". Indeed, both for extremal BHs attractors in $\mathscr{N}=2,\ d=4$ supergravity and for FV attractors in $\mathscr{N}=1,\ d=4$ supergravity (*at least* for the one coming from some peculiar compactifications of superstrings: See Sect. 4), there exist two different approaches to determining the attractors:

(i) the so-called *criticality conditions approach*, based on the direct solution of the conditions giving the stationary points of the relevant "effective potential";

(*ii*) the so-called *new attractor approach*, based on the solution of some fundamental geometrical identities evaluated along the criticality conditions of the relevant "effective potential".

Such two approaches are completely equivalent. Depending on the considered frameworks, it can be convenient to exploit one approach rather than the other (see e.g. [26] for an explicit case).

By using the relations (38), the $\mathcal{N}=2$ AEs (52) can be recast as follows [5]:

$$D_iV_{BH} = \partial_iV_{BH} = 2\overline{Z}D_iZ + iC_{ijk}g^{j\bar{l}}g^{k\bar{m}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{m}}\overline{Z}. \quad (53)$$

Equation (53) are nothing but the relations between the $\mathcal{N}=2$ central charge function Z (*graviphoton charge*) and the n_V *matter charges* Z_i (coming from the n_V Abelian vector supermultiplets), holding at the critical points of V_{BH} in the SK scalar manifold $\mathcal{M}_{n_V}$. As is seen, the tensor C_{ijk} plays a key role.

It is known that static, spherically symmetric, asymptotically flat extremal BHs in $d=4$ are described by an effective $d=1$ Lagrangian ([5], [105], and also [18] and [80]), with an effective scalar potential and effective fermionic "mass terms" terms controlled by the field-strength fluxes vector Q defined by (1). The *"apparent" gravitino mass* is given by Z, whereas the $n_V \times n_V$ *gaugino mass matrix* Λ_{ij} reads (see the second Ref. of [101, 102, 103])

$$\Lambda_{ij} = -iD_iZ_j = C_{ijk}g^{k\bar{k}}\overline{Z}_{\bar{k}} = \Lambda_{(ij)}. \quad (54)$$

Note that Λ_{ij} is part of the holomorphic/anti-holomorphic form of the $2n_V \times 2n_V$ covariant Hessian of Z, which is nothing but the holomorphic/anti-holomorphic form of the scalar mass matrix. The *supersymmetry order parameters*, related to the mixed gravitino-gaugino couplings, are given by the *matter charge (function)s* $D_iZ = Z_i$ (see the first of (38)).

By assuming that the Kähler potential is regular, i.e. that $|K| < \infty$ globally in $\mathcal{M}_{N_V}$ (or *at least* at the critical points of V_{BH}), one gets that

$$\partial_iV_{BH} = 0 \Leftrightarrow 2\overline{W}D_iW + ie^KW_{ijk}g^{j\bar{l}}g^{k\bar{m}}\left(\overline{D}_{\bar{l}}\overline{W}\right)\overline{D}_{\bar{m}}\overline{W} = 0. \quad (55)$$

3.1.1 Classification of Critical Points of V_{BH}

Starting from the general structure of the criticality conditions (55) and assuming also the *non-degeneracy* (i.e. $V_{BH}|_{\partial V_{BH}=0} > 0$) *condition*, the critical points of V_{BH} can be classified in three general classes, analyzed in the next three Subsubsubsects.

Supersymmetric ($\frac{1}{2}$-BPS)

The *supersymmetric* ($\frac{1}{2}$*-BPS*) critical points of V_{BH} are determined by the constraints (sufficient but not necessary conditions for (55))

$$Z \neq 0, D_i Z = 0, \forall i = 1, \ldots, n_V. \tag{56}$$

The horizon ADM squared mass at $\frac{1}{2}$-BPS critical points of V_{BH} saturates the BPS bound:

$$M^2_{ADM,H,\frac{1}{2}-BPS} = V_{BH,\frac{1}{2}-BPS} = \left[|Z|^2 + g^{i\bar{j}} (D_i Z) \overline{D}_{\bar{j}} \overline{Z} \right]_{\frac{1}{2}-BPS} = |Z|^2_{\frac{1}{2}-BPS} > 0. \tag{57}$$

Considering the $\mathcal{N} = 2,\ d = 4$ supergravity Lagrangian in a static, spherically symmetric, asymptotically flat extremal BH background, and denoting by ψ and λ^i, respectively, the gravitino and gaugino fields, it is easy to see that such a Lagrangian contains terms of the form (see the second and third Refs. of [101, 102, 103])

$$\begin{aligned} &Z\psi\psi; \\ &C_{ijk} g^{k\bar{k}} \left(\overline{D}_{\bar{k}} \overline{Z} \right) \lambda^i \lambda^j; \\ &(D_i Z) \lambda^i \psi. \end{aligned} \tag{58}$$

Thus, the conditions (56) imply the gaugino mass term and the $\lambda\psi$ term to vanish at the $\frac{1}{2}$-BPS critical points of V_{BH} in $\mathcal{M}_{n_V}$. It is interesting to remark that the gravitino "apparent mass" term $Z\psi\psi$ is in general non-vanishing, also when evaluated at the considered $\frac{1}{2}$-BPS attractors; this is ultimately a consequence of the fact that the extremal BH horizon geometry at the $\frac{1}{2}$-BPS (as well as at the non-BPS) attractors is Bertotti-Robinson $AdS_2 \times S^2$ [108, 109, 110].

Non-supersymmetric (non-BPS) with $Z \neq 0$

The *non-supersymmetric (*non-*BPS)* critical points of V_{BH} with non-vanishing central charge are determined by the constraints

$$Z \neq 0, D_i Z \neq 0, \textit{at least} \text{ for some } i \in \{1, \ldots, n_V\}, \tag{59}$$

which, substituted in (55), yield:

$$\begin{aligned} D_i Z &= -\frac{i}{2\overline{Z}} C_{ijk} g^{j\bar{l}} g^{k\bar{m}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \overline{D}_{\bar{m}} \overline{Z}, \quad \forall i = 1, \ldots, n_V; \\ &\Updownarrow \\ \overline{D}_{\bar{i}} \overline{Z} &= \frac{i}{2Z} \overline{C}_{\bar{i}\bar{j}\bar{k}} g^{l\bar{j}} g^{m\bar{k}} (D_l Z) D_m Z, \ \forall \bar{i} = \bar{1}, \ldots, \overline{n_V}, \end{aligned} \tag{60}$$

in turn implying that

$$\begin{aligned} g^{i\bar{i}} (D_i Z) \overline{D}_{\bar{i}} \overline{Z} &= -\frac{i}{2\overline{Z}} C_{ijk} g^{i\bar{i}} g^{j\bar{l}} g^{k\bar{m}} \left(\overline{D}_{\bar{i}} \overline{Z} \right) \left(\overline{D}_{\bar{l}} \overline{Z} \right) \overline{D}_{\bar{m}} \overline{Z} \\ &= \frac{i}{2Z} \overline{C}_{\bar{i}\bar{j}\bar{k}} g^{i\bar{i}} g^{l\bar{j}} g^{m\bar{k}} (D_i Z) (D_l Z) D_m Z. \end{aligned} \tag{61}$$

Such critical points are *non-supersymmetric* ones (i.e. they do *not* preserve any of the 8 supersymmetry degrees of freedom of the asymptotical Minkowski background), and they correspond to an extremal, non-BPS BH background. They are commonly named *non-BPS* $Z \neq 0$ *critical points of* V_{BH}.

AEs (55) and conditions (59) imply

$$\left(C_{ijk}\right)_{non-BPS,Z\neq0} \neq 0, \quad \text{for some } (i,j,k) \in \{1,\ldots,n_V\}^3. \tag{62}$$

By using (60) and the so-called SK geometry constraints (see the third of (39)), the horizon ADM squared mass corresponding to non-BPS $Z \neq 0$ critical points of V_{BH} can be elaborated as follows:

$$\begin{aligned} M^2_{ADM,H,non-BPS,Z\neq0} &= V_{BH,non-BPS,Z\neq0} = \left[|Z|^2 + g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{Z}\right]_{non-BPS,Z\neq0} \\ &= \left\{|Z|^2\left[1 + \frac{1}{4|Z|^4}R_{k\bar{r}n\bar{s}}g^{k\bar{m}}g^{t\bar{r}}g^{n\bar{l}}g^{u\bar{s}}\left(D_tZ\right)\left(D_uZ\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{m}}\overline{Z} + \right.\right. \\ &\left.\left. + \frac{1}{2|Z|^4}\left[g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{Z}\right]^2\right]\right\}_{non-BPS,Z\neq0}. \end{aligned} \tag{63}$$

As far as $g_{i\bar{j}}$ is strictly positive-definite globally (or *at least* at the non-BPS $Z \neq 0$ critical points of V_{BH}), $M^2_{ADM,H,non-BPS,Z\neq0}$ does *not* saturate the BPS bound ([9, 14, 16]):

$$\begin{aligned} M^2_{ADM,H,non-BPS,Z\neq0} &= V_{BH,non-BPS,Z\neq0} \\ &= \left[|Z|^2 + g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{Z}\right]_{non-BPS,Z\neq0} > |Z|^2_{non-BPS,Z\neq0}. \end{aligned} \tag{64}$$

Starting from (63), one can introduce and further elaborate the so-called *non-BPS* $Z \neq 0$ *supersymmetry breaking order parameter* as follows:

$$\begin{aligned} (0<)\,\mathscr{O}_{non-BPS,Z\neq0} &\equiv \left[\frac{g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{Z}}{|Z|^2}\right]_{non-BPS,Z\neq0} \\ &= -\left[\frac{i}{2\overline{Z}|Z|^2}C_{ijk}g^{i\bar{i}}g^{j\bar{l}}g^{k\bar{m}}\left(\overline{D}_{\bar{i}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{m}}\overline{Z}\right]_{non-BPS,Z\neq0} \\ &= \left[\frac{i}{2Z|Z|^2}\overline{C}_{\bar{i}\bar{j}\bar{k}}g^{i\bar{i}}g^{l\bar{j}}g^{m\bar{k}}\left(D_iZ\right)\left(D_lZ\right)D_mZ\right]_{non-BPS,Z\neq0}, \end{aligned} \tag{65}$$

where (61) were used. Since it holds that

$$\left[\frac{g^{i\bar{j}}(D_iZ)\overline{D}_{\bar{j}}\overline{Z}}{|Z|^2}\right]_{non-BPS,Z\neq 0} = \left\{\frac{1}{4|Z|^4}R_{k\bar{r}n\bar{s}}g^{k\overline{m}}g^{t\bar{r}}g^{n\bar{l}}g^{u\bar{s}}(D_tZ)(D_uZ)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z} + \frac{1}{2}\left[\frac{g^{i\bar{j}}(D_iZ)\overline{D}_{\bar{j}}\overline{Z}}{|Z|^2}\right]^2\right\}_{non-BPS,Z\neq 0}, \tag{66}$$

$\mathscr{O}_{non-BPS,Z\neq 0}$ defined by (65) can equivalently be rewritten as follows:

$$\mathscr{O}_{non-BPS,Z\neq 0} = \left[\frac{1}{4|Z|^4}g^{i\bar{j}}C_{ikn}\overline{C}_{\bar{j}\bar{r}\bar{s}}g^{n\bar{l}}g^{k\overline{m}}g^{t\bar{r}}g^{u\bar{s}}(D_tZ)(D_uZ)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}\right]_{non-BPS,Z\neq 0}. \tag{67}$$

Equation (67) imply that

$$M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = |Z|^2_{non-BPS,Z\neq 0}\left[1+\mathscr{O}_{non-BPS,Z\neq 0}\right] = \left\{|Z|^2\left[3-2\frac{\mathscr{R}(Z)}{g^{i\bar{j}}C_{ikn}\overline{C}_{\bar{j}\bar{r}\bar{s}}g^{n\bar{l}}g^{k\overline{m}}g^{t\bar{r}}g^{u\bar{s}}(D_tZ)(D_uZ)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}}\right]\right\}_{non-BPS,Z\neq 0}, \tag{68}$$

where the *sectional curvature* (see e.g. [111] and [112])

$$\mathscr{R}(Z) \equiv R_{\bar{i}\bar{j}k\bar{l}}g^{i\bar{i}}g^{j\bar{j}}g^{k\bar{k}}g^{l\bar{l}}(D_jZ)(D_lZ)\left(\overline{D}_{\bar{i}}\overline{Z}\right)\overline{D}_{\bar{k}}\overline{Z} \tag{69}$$

was introduced.

Now, by using the relations of SK geometry it is possible to show that

$$\begin{gathered}\overline{D}_{\overline{m}}D_iC_{jkl} = \left[\overline{D}_{\overline{m}},D_i\right]C_{jkl} = \overline{D}_{\overline{m}}D_{(i}C_{j)kl} = \overline{D}_{\overline{m}}D_{(i}C_{jkl)} \\ = 3C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\bar{n}\bar{p}\overline{m}} - 4g_{(l|\overline{m}}C_{|ijk)} \\ \Updownarrow \\ C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\bar{n}\bar{p}\overline{m}} = \frac{4}{3}g_{(l|\overline{m}}C_{|ijk)} + \overline{E}_{\overline{m}(ijkl)},\end{gathered} \tag{70}$$

where we introduced the rank-5 tensor

$$\overline{E}_{\overline{m}ijkl} = \overline{E}_{\overline{m}(ijkl)} \equiv \frac{1}{3}\overline{D}_{\overline{m}}D_iC_{jkl} = \frac{1}{3}\overline{D}_{\overline{m}}D_{(i}C_{jkl)} = C_{p(kl}C_{ij)n}g^{n\bar{n}}g^{p\bar{p}}\overline{C}_{\bar{n}\bar{p}\overline{m}} - \frac{4}{3}g_{(l|\overline{m}}C_{|ijk)} = g^{n\bar{n}}R_{(i|\overline{m}|j|\bar{n}}C_{n|kl)} + \frac{2}{3}g_{(i|\overline{m}}C_{|jkl)}, \tag{71}$$

where the SK geometry constraints were used, as well. Now, by recalling the criticality conditions (55) of V_{BH}, and by using (60), one gets that at non-BPS, $Z\neq 0$ critical points of V_{BH} it holds that

$$
\begin{aligned}
&2\overline{Z}D_iZ\\
&=\frac{i}{4Z^2}E_{i(\overline{s}\overline{n}\overline{t}\overline{u})}g^{p\overline{n}}g^{q\overline{s}}g^{r\overline{t}}g^{v\overline{u}}\left(D_pZ\right)\left(D_qZ\right)\left(D_rZ\right)D_vZ\\
&\quad+\frac{i}{3Z^2}\left(D_iZ\right)\overline{C}_{\overline{n}\overline{t}\overline{u}}g^{p\overline{n}}g^{r\overline{t}}g^{v\overline{u}}\left(D_pZ\right)\left(D_rZ\right)D_vZ.
\end{aligned}
\tag{72}
$$

By using (65), (72) and (65), with a little effort it is thus possible to compute that

$$
\begin{aligned}
&M^2_{ADM,H,non-BPS,Z\neq0}=V_{BH,non-BPS,Z\neq0}=|Z|^2_{non-BPS,Z\neq0}\left[1+\mathscr{O}_{non-BPS,Z\neq0}\right]\\
&=|Z|^2_{non-BPS,Z\neq0}\\
&\times\left\{4-\frac{3}{4}\left[\frac{1}{|Z|^2}\frac{E_{i\left(\overline{k}\overline{l}\overline{m}\overline{n}\right)}g^{i\overline{j}}g^{k\overline{k}}g^{l\overline{l}}g^{m\overline{m}}g^{n\overline{n}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(D_kZ\right)\left(D_lZ\right)\left(D_mZ\right)D_nZ}{N_3\left(Z\right)}\right]_{non-BPS,Z\neq0}\right\},
\end{aligned}
\tag{73}
$$

where we defined the complex cubic form

$$
N_3\left(Z\right)\equiv\overline{C}_{\overline{i}\overline{j}\overline{k}}g^{i\overline{i}}g^{j\overline{j}}g^{k\overline{k}}\left(D_iZ\right)\left(D_jZ\right)D_kZ.
\tag{74}
$$

Now, by comparing (73) with (48) and by recalling the definition (69), one obtains the following relations to hold at the non-BPS, $Z\neq0$ critical points of V_{BH}:

$$
\begin{aligned}
&\frac{3}{4}\left[\frac{1}{|Z|^2}\frac{E_{i\left(\overline{k}\overline{l}\overline{m}\overline{n}\right)}g^{i\overline{j}}g^{k\overline{k}}g^{l\overline{l}}g^{m\overline{m}}g^{n\overline{n}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(D_kZ\right)\left(D_lZ\right)\left(D_mZ\right)D_nZ}{N_3\left(Z\right)}\right]_{non-BPS,Z\neq0}-1\\
&=\left[\frac{\mathscr{R}\left(Z\right)}{2|Z|^2g^{t\overline{u}}\left(D_tZ\right)\overline{D}_{\overline{u}}\overline{Z}}\right]_{non-BPS,Z\neq0}\\
&=2\left[\frac{\mathscr{R}\left(Z\right)}{g^{i\overline{j}}C_{ikn}\overline{C}_{\overline{j}\overline{r}\overline{s}}g^{n\overline{l}}g^{k\overline{m}}g^{t\overline{r}}g^{u\overline{s}}\left(D_tZ\right)\left(D_uZ\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}}\right]_{non-BPS,Z\neq0}.
\end{aligned}
\tag{75}
$$

Let us now consider the case of homogeneous symmetric SK manifolds, in which the Kähler-invariant Riemann-Christoffel tensor $R_{i\overline{j}k\overline{l}}$ is covariantly constant[8]. From this it follows that [96]:

$$
D_mR_{i\overline{j}k\overline{l}}=0\Leftrightarrow D_iC_{jkl}=D_{(i}C_{j)kl}=0\Rightarrow\overline{D}_{\overline{m}}D_iC_{jkl}=0\Leftrightarrow D_m\overline{D}_{\overline{i}}\overline{C}_{\overline{j}\overline{k}\overline{l}}=0.
\tag{76}
$$

This implies the *global* vanishing of the tensor $\overline{E}_{\overline{i}jklm}$, yielding [96]

---

[8] Indeed, due to the reality of $R_{i\overline{j}k\overline{l}}$ in any Kähler manifold, it holds that

$$
D_mR_{i\overline{j}k\overline{l}}=0\Leftrightarrow\overline{D}_{\overline{m}}R_{i\overline{j}k\overline{l}}=0.
$$

$$C_{p(kl}C_{ij)n}g^{n\overline{n}}g^{p\overline{p}}\overline{C}_{\overline{npm}} = \frac{4}{3}g_{(l|\overline{m}}C_{|ijk)} \Leftrightarrow g^{n\overline{n}}R_{(i|\overline{m}|j|\overline{n}}C_{n|kl)} = -\frac{2}{3}g_{(i|\overline{m}}C_{|jkl)}. \quad (77)$$

By recalling (72) and (74), one obtains the following noteworthy relation, holding in homogeneous symmetric SK manifolds:

$$\left(Z|Z|^2\right)_{non-BPS,Z\neq 0} = \frac{i}{6}\left[N_3(Z)\right]_{non-BPS,Z\neq 0}, \quad (78)$$

implying that $\left[\frac{N_3(Z)}{Z}\right]_{non-BPS,Z\neq 0}$ has vanishing real part and strictly negative imaginary part, given by $-6|Z|^2_{non-BPS,Z\neq 0}$. By recalling (67), (78) implies the value of the *supersymmetry breaking order parameter* at non-BPS, $Z\neq 0$ critical points of V_{BH} in homogeneous symmetric SK manifolds to be

$$\mathscr{O}_{non-BPS,Z\neq 0} = 3 \Longrightarrow \Delta_{non-BPS,Z\neq 0} = 0. \quad (79)$$

By recalling (73), one thus finally gets that

$$M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = 4|Z|^2_{non-BPS,Z\neq 0} = \frac{2}{3}i\left[\frac{N_3(Z)}{Z}\right]_{non-BPS,Z\neq 0}, \quad (80)$$

where in the last step we used the relation (78). The result $V_{BH,non-BPS,Z\neq 0} = 4|Z|^2_{non-BPS,Z\neq 0}$ has been firstly obtained, by exploiting group-theoretical methods, in [21].

Finally, by recalling (75) and using (78) and (80), one obtains the following relation, holding for homogeneous symmetric SK manifolds:

$$\mathscr{R}(Z)|_{non-BPS,Z\neq 0} = -6|Z|^4_{non-BPS,Z\neq 0}. \quad (81)$$

It is worth pointing out that, while (76) (holding globally) are peculiar to homogeneous symmetric SK manifolds, (78), (79), (80), (81) hold in general also for homogeneous non-symmetric SK manifolds, in which the Riemann-Christoffel tensor $R_{i\overline{j}k\overline{l}}$ (and thus, through the SK constraints, C_{ijk}) is *not* covariantly constant. Indeed, as obtained in [28] for all the considered non-BPS, $Z\neq 0$ critical points of V_{BH} in homogeneous non-symmetric SK manifolds it holds that

$$\left[E_{i\left(\overline{klmn}\right)}g^{i\overline{j}}g^{k\overline{k}}g^{l\overline{l}}g^{m\overline{m}}g^{n\overline{n}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)(D_kZ)(D_lZ)(D_mZ)D_nZ\right]_{non-BPS,Z\neq 0} = 0, \quad (82)$$

which actually seems to be the most general (necessary and sufficient) condition in order for (78), (79), (80), (81) to hold. Finally, it should be stressed that in [10] the result (79) and thus $V_{BH,non-BPS,Z\neq 0} = 4|Z|^2_{non-BPS,Z\neq 0}$ was obtained for a generic SK geometry with a cubic holomorphic prepotential (corresponding to the large volume limit of Type IIA on Calabi-Yau threefolds), at least for the non-BPS, $Z\neq 0$ critical points of V_{BH} satisfying the Ansatz

$$z^i_{non-BPS,Z\neq 0} = p^i t(p,q), \ \forall i = 1,\ldots,n_V, \tag{83}$$

where the $z^i_{non-BPS,Z\neq 0}$s are the critical moduli, and $t(p,q)$ is a purely charge-dependent quantity.

Furthermore, it is worth noticing that the general criticality conditions (52) of V_{BH} can be recognized to be the general Ward identities relating the gravitino mass Z, the gaugino masses D_iD_jZ and the supersymmetry-breaking order parameters D_iZ in a generic spontaneously broken supergravity theory [113, 114, 115]. Indeed, away from $\frac{1}{2}$-BPS critical points (i.e. for $D_iZ \neq 0$ for some i), the AEs (52) can be re-expressed as follows (see also [32]):

$$\left(\mathbf{M}_{ij}h^j\right)_{\partial V_{BH}=0} = 0, \tag{84}$$

with

$$\mathbf{M}_{ij} \equiv D_iD_jZ + 2\frac{\overline{Z}}{\left[g^{k\bar{k}}(D_kZ)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\right]}(D_iZ)(D_jZ), (\text{ Kähler weights}\,(1,-1)), \tag{85}$$

and

$$h^j \equiv g^{j\bar{j}}\overline{D}_{\bar{j}}\overline{Z}, (\text{Kähler weights}\,(-1,1)). \tag{86}$$

For a non-vanishing contravariant vector h^j (i.e. away from $\frac{1}{2}$-BPS critical points, as pointed out above), (84) admits a solution iff the $n_V \times n_V$ complex symmetric matrix $\mathbf{M}_{ij}$ has vanishing determinant (implying that it has at most $n_V - 1$ non-vanishing eigenvalues) at the considered (non-BPS) critical points of V_{BH} (however, notice that $\mathbf{M}_{ij}$ is symmetric but not necessarily Hermitian, and thus in general its eigenvalues are not necessarily real). Such a reasoning holds for all non-BPS critical points of V_{BH}, i.e. for the classes II and III of the presented classification.

In general, non-BPS $Z \neq 0$ critical points of V_{BH} in $\mathscr{M}_{n_V}$ are not necessarily stable, because the $2n_V \times 2n_V$ (covariant) Hessian matrix (in $(z,\bar{z})$-coordinates) of V_{BH} evaluated at such points is not necessarily strictly positive-definite. An explicit condition of stability of non-BPS $Z \neq 0$ critical points of V_{BH} can be worked out in the $n_V = 1$ case (see [17, 18, 26]).

In general, (58) and conditions (59) imply the gaugino mass term, the $\lambda\psi$ term and the gravitino “apparent mass” term $Z\psi\psi$ to be non-vanishing, when evaluated at the considered non-BPS $Z \neq 0$ critical points of V_{BH}.

Non-supersymmetric (non-BPS) with $Z = 0$

The *non-supersymmetric (non-BPS)* critical points of V_{BH} with vanishing central charge are determined by the constraints

$$Z = 0, \ D_iZ \overset{Z=0}{=} \partial_iZ \neq 0, \ \textit{at least} \text{ for some } i \in \{1,\ldots,n_V\}, \tag{87}$$

which, substituted in (55), yield:

$$C_{ijk}g^{j\bar{l}}g^{k\bar{m}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{m}}\overline{Z}\overset{Z=0}{=}C_{ijk}g^{j\bar{l}}g^{k\bar{m}}\left(\overline{\partial}_{\bar{l}}\overline{Z}\right)\overline{\partial}_{\bar{m}}\overline{Z}=0,\ \forall i=1,\ldots,n_V. \tag{88}$$

Such critical points are *non-supersymmetric* ones, but, differently from the class II considered above, they correspond to an extremal, non-BPS BH background in which the horizon $\mathcal{N}=2$, $d=4$ supersymmetry algebra is not centrally extended. They are commonly named *non-BPS* $Z=0$ *critical points of* V_{BH}.

The horizon ADM squared mass corresponding to non-BPS $Z=0$ critical points of V_{BH} does *not* saturate the BPS bound ([9, 14, 16]):

$$\begin{aligned}&M^2_{ADM,H,non-BPS,Z=0}=V_{BH,non-BPS,Z=0}\\&=\left\{g^{i\bar{j}}\left(\partial_i Z\right)\overline{\partial}_{\bar{j}}\overline{Z}\right\}_{non-BPS,Z=0}>\left(|Z|^2\right)_{non-BPS,Z=0}=0,\end{aligned} \tag{89}$$

as far as $g_{i\bar{j}}$ is strictly positive-definite globally (or *at least* at the considered critical points of V_{BH}). Equation (88) suggest the following sub-classification of non-BPS $Z=0$ critical points of V_{BH}:

(III.1) Critical points determined by the conditions

$$\begin{cases}Z=0,\\ D_iZ\overset{Z=0}{=}\partial_iZ\neq 0, \textit{at least} \text{ for some } i\in\{1,\ldots,n_V\},\\ C_{ijk}=0,\forall i,j,k,\end{cases} \tag{90}$$

directly solving (88) and thus AEs (55). This is the only possible case for $n_V=1$.

In particular, non-BPS $Z=0$ critical points of V_{BH} do not exist at all in the $n_V=1$ case of the so-called "*d-SK geometries*", whose stringy origin is e.g. Type IIA on CY_3 in the large volume limit of CY_3 (see e.g. [10]). Indeed, in such a case in special projective coordinates (with Kähler gauge fixed such that $X^0\equiv 1$) the holomorphic prepotential $\mathcal{F}$ and W_{ijk}, respectively, read

$$\begin{aligned}&\mathcal{F}=d_{ijk}z^iz^jz^k;\\&C_{ijk}=e^Kd_{ijk},\end{aligned} \tag{91}$$

and thus, for $|K|<\infty$ at least at the considered critical points of V_{BH}, the third of conditions (90) cannot be satisfied.

(III.2) Critical points determined by the conditions

$$\begin{cases}Z=0,\\ D_iZ\overset{Z=0}{=}\partial_iZ\neq 0,\ \textit{at least} \text{ for some } i\in\{1,\ldots,n_V\},\\ C_{ijk}\neq 0,\ \textit{at least} \text{ for some } (i,j,k)\in\{1,\ldots,n_V\}^3,\end{cases} \tag{92}$$

for which (88), and thus AEs (55), are not trivially solved.

In general, non-BPS $Z = 0$ critical points of V_{BH} in $\mathscr{M}_{n_V}$ are not necessarily stable, because the $2n_V \times 2n_V$ (covariant) Hessian matrix (in $(z, \bar{z})$-coordinates) of V_{BH} evaluated at such points is not necessarily strictly positive-definite. An explicit condition of stability of non-BPS $Z = 0$ critical points of V_{BH} can be worked out in the $n_V = 1$ case [26].

In general, (58) and conditions (56) imply the $\lambda\psi$ term to be non-vanishing and the gravitino "apparent mass" term $Z\psi\psi$ to vanish, when evaluated at the considered non-BPS $Z = 0$ critical points of V_{BH}, characterized by vanishing (class III.1) or non-vanishing (class III.2) gaugino mass terms.

Non-BPS $Z = 0$ attractors in the so-called st^2 and stu models [116, 117] have been recently studied in [43], and their relation with the $\frac{1}{2}$-BPS attractors has been analyzed in light of the uplift to $\mathscr{N} = 8,\ d = 4$ supergravity.

3.2 Stability of Critical Points of V_{BH}

3.2.1 n_V-Moduli

In order to decide whether a critical point of V_{BH} is an attractor in strict sense, one has to consider the following condition:

$$H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}} \equiv D_a D_b V_{BH} > 0 \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial \phi^c} = 0 \quad \forall c = 1, \ldots, 2n_V, \tag{93}$$

i.e. the condition of (strict) positive-definiteness of the real $2n_V \times 2n_V$ Hessian matrix $H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}}$ of V_{BH} (which is nothing but the squared mass matrix of the moduli) at the critical points of V_{BH}, expressed in the real parameterization through the ϕ-coordinates. Since V_{BH} is positive-definite, a stable critical point (namely, an attractor in strict sense) is necessarily a(n at least local) minimum, and therefore it fulfills the condition (93).

In general, $H_{\mathbb{R}}^{V_{BH}}$ may be block-decomposed in $n_V \times n_V$ real matrices:

$$H_{\mathbb{R}}^{V_{BH}} = \begin{pmatrix} \mathscr{A} & \mathscr{C} \\ \mathscr{C}^T & \mathscr{B} \end{pmatrix}, \tag{94}$$

with $\mathscr{A}$ and $\mathscr{B}$ being $n_V \times n_V$ real symmetric matrices:

$$\mathscr{A}^T = \mathscr{A}, \mathscr{B}^T = \mathscr{B} \Leftrightarrow \left(H_{\mathbb{R}}^{V_{BH}}\right)^T = H_{\mathbb{R}}^{V_{BH}}. \tag{95}$$

In the local complex $(z, \bar{z})$-parameterization, the $2n_V \times 2n_V$ Hessian matrix of V_{BH} reads

$$H_{\mathbb{C}}^{V_{BH}} \equiv H_{\widehat{i}\widehat{j}}^{V_{BH}} \equiv \begin{pmatrix} D_i D_j V_{BH} & D_i \overline{D}_{\bar{j}} V_{BH} \\ D_j \overline{D}_{\bar{i}} V_{BH} & \overline{D}_{\bar{i}} \overline{D}_{\bar{j}} V_{BH} \end{pmatrix} = \begin{pmatrix} \mathscr{M}_{ij} & \mathscr{N}_{i\bar{j}} \\ \overline{\mathscr{N}_{i\bar{j}}} & \overline{\mathscr{M}_{ij}} \end{pmatrix}, \tag{96}$$

where the hatted indices $\hat{\imath}$ and $\hat{\jmath}$ may be holomorphic or antiholomorphic. $H_{\mathbb{C}}^{V_{BH}}$ is the matrix actually computable in the SKG formalism presented in Sect. 2 (see below, (98) and (99)).

In general, $\frac{1}{2}$-BPS critical points are (at least local) minima of V_{BH}, and therefore they are stable; thus, they are *attractors* in strict sense. Indeed, the $2n_V \times 2n_V$ (covariant) Hessian matrix $H_{\mathbb{C}}^{V_{BH}}$ evaluated at such points is strictly positive-definite [5]:

$$\begin{aligned}
(D_i D_j V_{BH})_{\frac{1}{2}-BPS} &= (\partial_i \partial_j V_{BH})_{\frac{1}{2}-BPS} = 0, \\
\left(D_i \overline{D}_{\bar{j}} V_{BH}\right)_{\frac{1}{2}-BPS} &= \left(\partial_i \overline{\partial}_{\bar{j}} V_{BH}\right)_{\frac{1}{2}-BPS} \\
&= 2\left(g_{i\bar{j}} V_{BH}\right)_{\frac{1}{2}-BPS} = 2\, g_{i\bar{j}}\Big|_{\frac{1}{2}-BPS} |Z|^2_{\frac{1}{2}-BPS} > 0,
\end{aligned} \tag{97}$$

where here and below the notation "> 0" ("< 0") is understood as strict positive-(negative-)definiteness. The Hermiticity and (strict) positive-definiteness of the (covariant) Hessian matrix $H_{\mathbb{C}}^{V_{BH}}$ at the $\frac{1}{2}$-BPS critical points are due to the Hermiticity and – assumed – (strict) positive-definiteness (actually holding globally) of the metric $g_{i\bar{j}}$ of the SK scalar manifold being considered.

On the other hand, non-BPS critical points of V_{BH} does not automatically fulfill the condition (93), and a more detailed analysis [21, 18] is needed.

Using the properties of SKG, one obtains:

$$\mathscr{M}_{ij} \equiv D_i D_j V_{BH} = D_j D_i V_{BH} = 4i\overline{Z} C_{ijk} g^{k\bar{k}} \left(\overline{D}_{\bar{k}} \overline{Z}\right) + i\left(D_j C_{ikl}\right) g^{k\bar{k}} g^{l\bar{l}} \left(\overline{D}_{\bar{k}} \overline{Z}\right)\left(\overline{D}_{\bar{l}} \overline{Z}\right); \tag{98}$$

$$\begin{aligned}
\mathscr{N}_{i\bar{j}} \equiv\ & D_i \overline{D}_{\bar{j}} V_{BH} = \overline{D}_{\bar{j}} D_i V_{BH} = 2\Big[g_{i\bar{j}} |Z|^2 + (D_i Z)\left(\overline{D}_{\bar{j}} \overline{Z}\right) \\
& + g^{l\bar{n}} C_{ikl} \overline{C}_{\bar{j}\bar{m}\bar{n}} g^{k\bar{k}} g^{m\bar{m}} \left(\overline{D}_{\bar{k}} \overline{Z}\right)(D_m Z)\Big],
\end{aligned} \tag{99}$$

with $D_j C_{ikl}$ given by (40). Clearly, evaluating (98) and (99) constrained by the $\frac{1}{2}$-BPS conditions $D_i Z = 0, \forall i = 1, \ldots, n_V$, one reobtains the results (97). Here, we limit ourselves to point out that further noteworthy elaborations of $\mathscr{M}_{ij}$ and $\mathscr{N}_{i\bar{j}}$ can be performed in homogeneous symmetric SK manifolds, where $D_j C_{ikl} = 0$ globally [21], and that the Kähler-invariant (2,2)-tensor $g^{l\bar{n}} C_{ikl} \overline{C}_{\bar{j}\bar{m}\bar{n}}$ can be rewritten in terms of the Riemann-Christoffel tensor $R_{i\bar{j}k\bar{m}}$ by using the so-called "SKG constraints" (see the third of (39)) [18]. Moreover, the differential Bianchi identities for $R_{i\bar{j}k\bar{m}}$ imply $\mathscr{M}_{ij}$ to be symmetric (see comment below (39) and (40)).

Thus, one gets the following global properties:

$$\mathscr{M}^T = \mathscr{M},\ \mathscr{N}^\dagger = \mathscr{N} \Leftrightarrow \left(H_{\mathbb{C}}^{V_{BH}}\right)^T = H_{\mathbb{C}}^{V_{BH}}, \tag{100}$$

implying that

$$\left(H_{\mathbb{C}}^{V_{BH}}\right)^\dagger = H_{\mathbb{C}}^{V_{BH}} \Leftrightarrow \mathscr{M}^\dagger = \mathscr{M},\ \mathscr{N}^T = \mathscr{N} \Leftrightarrow \overline{\mathscr{M}} = \mathscr{M},\ \overline{\mathscr{N}} = \mathscr{N}. \tag{101}$$

It should be stressed clearly that the symmetry but non-Hermiticity of $H_{\mathbb{C}}^{V_{BH}}$ actually does not matter, because what one is interested in are the eigenvalues of the real form $H_{\mathbb{R}}^{V_{BH}}$, which is real and symmetric, and therefore admitting $2n_V$ *real* eigenvalues.

The relation between $H_{\mathbb{R}}^{V_{BH}}$ expressed by (94) and $H_{\mathbb{C}}^{V_{BH}}$ given by (96) is expressed by the following relations between the $n_V \times n_V$ sub-blocks of $H_{\mathbb{R}}^{V_{BH}}$ and $H_{\mathbb{C}}^{V_{BH}}$ [17, 29]:

$$\begin{cases} \mathscr{M} = \frac{1}{2}\left(\mathscr{A} - \mathscr{B}\right) + \frac{i}{2}\left(\mathscr{C} + \mathscr{C}^T\right); \\ \mathscr{N} = \frac{1}{2}\left(\mathscr{A} + \mathscr{B}\right) + \frac{i}{2}\left(\mathscr{C}^T - \mathscr{C}\right), \end{cases} \tag{102}$$

or its inverse

$$\begin{cases} \mathscr{A} = Re\mathscr{M} + Re\mathscr{N}; \\ \mathscr{B} = Re\mathscr{N} - Re\mathscr{M}; \\ \mathscr{C} = Im\mathscr{M} - Im\mathscr{N}. \end{cases} \tag{103}$$

The structure of the Hessian matrix gets simplified at the critical points of V_{BH}, because the covariant derivatives may be substituted by the flat ones; the critical Hessian matrices in complex holomorphic/antiholomorphic and real local parameterizations, respectively, read

$$H_{\mathbb{C}}^{V_{BH}}\Big|_{\partial V_{BH}=0} \equiv \begin{pmatrix} \partial_i \partial_j V_{BH} & \partial_i \overline{\partial}_{\overline{j}} V_{BH} \\ \partial_j \overline{\partial}_{\overline{i}} V_{BH} & \overline{\partial}_{\overline{i}} \overline{\partial}_{\overline{j}} V_{BH} \end{pmatrix}_{\partial V_{BH}=0} = \begin{pmatrix} \mathscr{M} & \mathscr{N} \\ \overline{\mathscr{N}} & \overline{\mathscr{M}} \end{pmatrix}_{\partial V_{BH}=0}; \tag{104}$$

$$H_{\mathbb{R}}^{V_{BH}}\Big|_{\partial V_{BH}=0} = \frac{\partial^2 V_{BH}}{\partial \phi^a \partial \phi^b}\Big|_{\partial V_{BH}=0} = \begin{pmatrix} \mathscr{A} & \mathscr{C} \\ \mathscr{C}^T & \mathscr{B} \end{pmatrix}_{\partial V_{BH}=0}. \tag{105}$$

Thus, the study of the condition (93) finally amounts to the study of the *eigenvalue problem* of the real symmetric $2n_V \times 2n_V$ critical Hessian matrix $H_{\mathbb{R}}^{V_{BH}}\Big|_{\partial V_{BH}=0}$ given by (105), which is the Cayley-transformed of the complex (symmetric, but not necessarily Hermitian) $2n_V \times 2n_V$ critical Hessian $H_{\mathbb{C}}^{V_{BH}}\Big|_{\partial V_{BH}=0}$ given by (104).

3.2.2 1-Modulus

Once again, the situation strongly simplifies in $n_V = 1$ SKG.

Indeed, for $n_V = 1$ the moduli-dependent matrices $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$, and $\mathscr{N}$ introduced above are simply scalar functions. In particular, $\mathscr{N}$ is real, since $\mathscr{C}$ trivially satisfies $\mathscr{C} = \mathscr{C}^T$. The stability condition (93) can thus be written as

$$H_{\mathbb{R}}^{V_{BH}} \equiv D_a D_b V_{BH} > 0, (a, b = 1, 2) \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial \phi^c} = 0 \quad \forall c = 1, 2, \tag{106}$$

and (98) and (99) respectively simplify to

$$\mathscr{M} \equiv D^2 V_{BH} = 4i\overline{Z}Cg^{-1}\overline{DZ} + i\,(DC)\,g^{-2}\left(\overline{DZ}\right)^2; \tag{107}$$

$$\mathscr{N} \equiv D\overline{D}V_{BH} = \overline{D}DV_{BH} = 2\left[g\,|Z|^2 + |DZ|^2 + |C|^2 g^{-3}\,|DZ|^2\right], \tag{108}$$

DC being given by the case $n_V = 1$ of (40):

$$\begin{aligned} DC &= \partial C + [(\partial K) + 3\Gamma]\,C = \partial C + [(\partial K) - 3\partial ln\,(g)]\,C \\ &= \left\{\partial + \left[\partial ln\left(\frac{e^K}{\left(\overline{\partial}\partial K\right)^3}\right)\right]\right\}C, \end{aligned} \tag{109}$$

where the $n_V = 1$ Christoffel connection

$$\Gamma = -g^{-1}\partial g = -\partial ln\,(g) \tag{110}$$

was used. It is easy to show that the stability condition (106) for critical points of V_{BH} in $n_V = 1$ SKG can be equivalently reformulated as the strict bound

$$\mathscr{N}\,|_{\partial V_{BH}=0} > |\mathscr{M}|_{\partial V_{BH}=0}\,. \tag{111}$$

Let us now see how such a bound can be further elaborated for the three possible classes of critical points of V_{BH}.

$\frac{1}{2}$-BPS

$$\mathscr{M}_{\frac{1}{2}-BPS} \equiv D^2 V_{BH}\big|_{\frac{1}{2}-BPS} = \left[3\overline{Z}D^2 Z + g^{-1}\left(D^3 Z\right)\overline{DZ}\right]_{\frac{1}{2}-BPS} = 0; \tag{112}$$

$$\mathscr{N}_{\frac{1}{2}-BPS} \equiv D\overline{D}V_{BH}\big|_{\frac{1}{2}-BPS} = \left[2g\,|Z|^2 + g^{-1}\left|D^2 Z\right|^2\right]_{\frac{1}{2}-BPS} = 2\left(g\,|Z|^2\right)_{\frac{1}{2}-BPS}. \tag{113}$$

Equations (112) and (113) are nothing but the 1-modulus case of (97), and they directly satisfy the bound (111). Thus, consistently with that stated above, the $\frac{1}{2}$-BPS class of critical points of V_{BH} actually is a class of attractors in strict sense (*at least* local minima of V_{BH}).

Non-BPS, $Z \neq 0$

$$\begin{aligned}
\mathcal{M}_{non-BPS,Z\neq0} &\equiv D^2 V_{BH}\Big|_{non-BPS,Z\neq0} \\
&= -2\left\{ g^{-1}\overline{Z}DZ\left[g^{-2}\left|C\right|^2 Dln\left(Z\right) + gDln\left(C\right)\right]\right\}_{non-BPS,Z\neq0} \quad (114)\\
&= i\left\{ Cg^{-3}\left(\overline{DZ}\right)^2\left[g^{-2}\left|C\right|^2 Dln\left(Z\right) + gDln\left(C\right)\right]\right\}_{non-BPS,Z\neq0};
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{non-BPS,Z\neq0} &\equiv D\overline{D}V_{BH}\Big|_{non-BPS,Z\neq0} = \overline{D}DV_{BH}\Big|_{non-BPS,Z\neq0} \\
&= 2\left\{ \left|DZ\right|^2\left[1 + \frac{5}{4}g^{-3}\left|C\right|^2\right]\right\}_{non-BPS,Z\neq0}.
\end{aligned} \quad (115)$$

Equation (114) yields that

$$\begin{aligned}
&\left|\mathcal{M}\right|^2_{non-BPS,Z\neq0} \\
&= 4\left\{ \left|DZ\right|^4\left[\left|C\right|^4 g^{-6} + \frac{1}{4}g^{-4}\left|DC\right|^2 + 2g^{-3}Re\left[C\left(\overline{DC}\right)Dln\left(Z\right)\right]\right]\right\}_{non-BPS,Z\neq0}.
\end{aligned} \quad (116)$$

By substituting (115) and (116) into the strict inequality (111), one finally obtains the stability condition for non-BPS, $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG [17]:

$$\mathcal{N}_{non-BPS,Z\neq0} > \left|\mathcal{M}\right|_{non-BPS,Z\neq0}; \quad (117)$$

$$\Updownarrow$$

$$\begin{aligned}
&1 + \frac{5}{4}\left(\left|C\right|^2 g^{-3}\right)_{non-BPS,Z\neq0} \\
&> \sqrt{\left[\left|C\right|^4 g^{-6} + \frac{1}{4}g^{-4}\left|DC\right|^2 + 2g^{-3}Re\left[C\left(\overline{DC}\right)\left(\overline{D}ln\overline{Z}\right)\right]\right]_{non-BPS,Z\neq0}}.
\end{aligned} \quad (118)$$

As is seen from such a condition, in general $(DC)_{non-BPS,Z\neq0}$ is the fundamental geometrical quantity playing a key role in determining the stability of non-BPS, $Z \neq 0$ critical points of V_{BH} in 1-modulus SK geometry.

Non-BPS, $Z = 0$

$$\mathcal{M}_{non-BPS,Z=0} \equiv D^2 V_{BH}\Big|_{non-BPS,Z=0} = i\left[g^{-2}\left(\partial C\right)\left(\overline{\partial}\overline{Z}\right)^2\right]_{non-BPS,Z=0}; \quad (119)$$

$$\mathcal{N}_{non-BPS,Z=0} \equiv D\overline{D}V_{BH}\Big|_{non-BPS,Z=0} = 2\left|\partial Z\right|^2_{non-BPS,Z=0}, \quad (120)$$

where (92) and (90) have been used. Equation (119) yields that

$$|\mathscr{M}|_{non-BPS,Z=0} = \left[g^{-2}\,|\partial C|\,|\partial Z|^2\right]_{non-BPS,Z=0}. \tag{121}$$

By substituting (120) and (121) into the strict inequality (111), one finally obtains the stability condition for non-BPS, $Z = 0$ critical points of V_{BH} in $n_V = 1$ SKG:

$$\mathscr{N}_{non-BPS,Z=0} > |\mathscr{M}|_{non-BPS,Z=0}; \tag{122}$$

$$\Updownarrow$$

$$2g^2_{non-BPS,Z=0} > |\partial C|_{non-BPS,Z=0}. \tag{123}$$

Even though the stability condition (123) have been obtained by correctly using (92) and (90), holding at the non-BPS, $Z = 0$ critical points of V_{BH}, in some cases it may happen that, in the limit of approaching the non-BPS, $Z = 0$ critical point of V_{BH}, in DC (given by (109)) the "connection term" $[(\partial K) + 3\Gamma]\,C$ is not necessarily subleading with respect to the "differential term" ∂C. Thus, the condition (123) can be rewritten as follows:

$$\begin{aligned} 2\left(\overline{\partial}\partial K\right)^2_{non-BPS,Z=0} &> \left|\left\{\partial + \left[(\partial K) - 3\partial ln\,(g)\right]\right\} C\right|_{non-BPS,Z=0} \\ &= \left|\left\{\partial + \left[\partial ln\left(\frac{e^K}{\left(\overline{\partial}\partial K\right)^3}\right)\right]\right\} C\right|_{non-BPS,Z=0}. \end{aligned} \tag{124}$$

Remark

Let us consider the 1-modulus stability conditions (117) and (123) and (124). It is immediate to realize that they are both satisfied when the function C is globally covariantly constant:

$$DC = \partial C + [(\partial K) + 3\Gamma]\,C = 0, \tag{125}$$

i.e. for the so-called homogeneous symmetric $(dim_{\mathbb{C}} = n_V = 1)$ SK geometry [95, 96], univoquely associated to the coset manifold $\frac{SU(1,1)}{U(1)}$. Such a SK manifold can be twofold characterized as:

(i) the $n = 0$ element of the irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$ (with $n_V = n + rank = n + 1$), or equivalently the $n = -2$ element of the reducible rank-3 infinite sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$ (with $n_V = n + rank = n + 3$). In such a case, $\frac{SU(1,1)}{U(1)}$ is endowed with a quadratic holomorphic prepotential function reading (in a suitable projective special coordinate, with Kähler gauge fixed such that $X^0 = 1$; see [21] and Refs. therein)

$$\mathscr{F}(z) = \frac{i}{2}\left(z^2 - 1\right). \tag{126}$$

By recalling the first of (39), such a prepotential yields $C=0$ globally (and thus (125)), and, therefore, by using the SKG constraints (i.e. the third of (39)) it yields also the constant scalar curvature

$$\mathscr{R} \equiv g^{-2}R = -2, \tag{127}$$

where $R \equiv R_{1\bar{1}1\bar{1}}$ denotes the unique component of the Riemann tensor. As obtained in [21], quadratic (homogeneous symmetric) SK geometries only admit $\frac{1}{2}$-BPS and non-BPS, $Z=0$ critical points of V_{BH}. Thus, it can be concluded that the 1-dim. quadratic SK geometry determined by the prepotential (126) admits *all* stable critical points of V_{BH}.

(*ii*) the rank-1 $s=t=u\equiv z$ *degeneration* of the so-called *stu* model [116, 117] ($n=0$ element of the reducible rank-3 infinite sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$), or equivalently the rank-1 $s=t\equiv z$ *degeneration* of the so-called st^2 model ($n=-1$ element of $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$), or also as an isolated case in the classification of homogeneous symmetric SK manifolds (see e.g. [118]). In such a case, $\frac{SU(1,1)}{U(1)}$ is endowed with a cubic holomorphic prepotential function reading (in a suitable projective special coordinate, with Kähler gauge fixed such that $X^0=1$; see e.g. [21] and Refs. therein)

$$\mathscr{F}(z) = \rho z^3,\ \rho \in \mathbb{C}, \tag{128}$$

constrained by the condition $Im(z)<0$. It admits an uplift to *pure* $\mathscr{N}=2$ supergravity in $d=5$. By recalling the first of (39), such a prepotential yields $C=6\rho e^K$ (and thus (125)), and consequently it also yields the constant scalar curvature

$$\mathscr{R} \equiv g^{-2}R = g^{-2}\left(-2g^2 + g^{-1}|C|^2\right) = -\frac{2}{3}, \tag{129}$$

where the SKG constraints (i.e. the third of (39)) and the global value[9] $|C|^2 g^{-3} = \frac{4}{3}$ have been used. As it can be computed (see e.g. [32]), the 1-dim. SK geometry determined by the prepotential (128) admits, beside the (stable) $\frac{1}{2}$-BPS ones, stable non-BPS $Z\neq 0$ critical points of V_{BH}. Thus, it is another example in which *all* critical points of V_{BH} actually are attractors in a strict sense.

Clearly, the quadratic and cubic homogeneous symmetric 1-modulus SK geometries (respectively determined by holomorphic prepotentials (126) and (128)) are not the only ones (with $n_V=1$) admitting stable non-BPS critical points of V_{BH}. For instance, as studied in [26], the 1-modulus SK geometries of the moduli space of the (mirror) Fermat CY_3 *quintic* $\mathscr{M}_5'$ and *octic* $\mathscr{M}_8'$ admit, in a suitable neighbourhood of the LG point, stable non-BPS $(Z\neq 0)$ critical points of V_{BH}.

It is worth remarking that recent works [33, 36, 46] gave a complete treatment of the issue of stability of non-BPS attractors in the framework of homogeneous

[9] The global value $|C|^2 g^{-3} = \frac{4}{3}$ for homogeneous symmetric cubic $n_V=1$ SK geometries is yielded by the $n_V=1$ case of (77).

SKGs, finding that the massless modes of the non-BPS Hessian matrix actually are "flat directions" of V_{BH} at the considered class of critical points. This means that non-BPS attractors in $\mathcal{N}=2$, $d=4$ supergravity have a related moduli space, spanned by those moduli which are not stabilized at the BH horizon. However, it should be pointed out that such an emergence of moduli spaces do not violate the Attractor Mechanism and/or the determinacy of BH thermodinamical properties, because the non-BPS BH entropy simply does not depend on the scalar degrees of freedom spanning the moduli space of the considered class ($Z \neq 0$ or $Z=0$) of non-BPS critical points of V_{BH}. Such considerations hold also for $\mathcal{N}>2$-extended, $d=4$ supergravities (where also BPS attractors can have a related moduli space), and in general in all theories with a homogeneous (not necessarily symmetric) scalar manifold [33, 40, 46, 49].

3.3 $\mathcal{N}$ = 2, d = 4 *General Formulation*

3.3.1 Special Kähler Geometry Identities

We will now derive some important identities of the SK geometry [9, 14, 17, 18, 23, 119] of the scalar manifold of $\mathcal{N}=2$, $d=4$ ungauged supergravity. Such identities extend the results obtained by Ferrara and Kallosh in [3].

Let us start by considering the covariant antiholomorphic derivative of $\overline{Z}$; by recalling the definition (23) and using the second of *Ansätze* (42), one gets

$$\overline{D}_{\overline{j}}\overline{Z} = q_{\Lambda}\overline{D}_{\overline{i}}\overline{L}^{\Lambda} - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}\overline{D}_{\overline{j}}\overline{L}^{\Delta}. \tag{130}$$

The contraction of both sides with $g^{i\overline{j}}D_iL^{\Sigma}$ then yields

$$g^{i\overline{j}}\left(D_iL^{\Sigma}\right)\overline{D}_{\overline{j}}\overline{Z} = q_{\Lambda}g^{i\overline{j}}\left(D_iL^{\Sigma}\right)\overline{D}_{\overline{j}}\overline{L}^{\Lambda} - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}g^{i\overline{j}}\left(D_iL^{\Sigma}\right)\overline{D}_{\overline{j}}\overline{L}^{\Delta}. \tag{131}$$

By exploiting the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse (see (44) and (167) further below, as well), recalling the first of the Ansätze (42), and using the result of SK geometry (see e.g. [64])

$$g^{i\overline{j}}\left(D_iL^{\Lambda}\right)\overline{D}_{\overline{j}}\overline{L}^{\Sigma} = -\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Lambda\Sigma} - \overline{L}^{\Lambda}L^{\Sigma}, \tag{132}$$

Equation (131) can be further elaborated as follows:

$$\begin{aligned}
g^{i\overline{j}}\left(D_iL^{\Sigma}\right)\overline{D}_{\overline{j}}\overline{Z} &= q_{\Lambda}\left[-\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda} - \overline{L}^{\Sigma}L^{\Lambda}\right] - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}\left[-\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta} - \overline{L}^{\Sigma}L^{\Delta}\right]\\
&= -\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda}q_{\Lambda} - \overline{L}^{\Sigma}\left(L^{\Lambda}q_{\Lambda} - M_{\Lambda}p^{\Lambda}\right) + \frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta}\left(Re\mathcal{N}_{\Delta\Lambda}\right)p^{\Lambda} + \frac{i}{2}p^{\Sigma}\\
&= \frac{i}{2}p^{\Sigma} - \overline{L}^{\Sigma}Z + \frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta}\left(Re\mathcal{N}_{\Delta\Lambda}\right)p^{\Lambda} - \frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda}q_{\Lambda}.
\end{aligned} \tag{133}$$

Now, by subtracting to the expression (133) its complex conjugate, one gets

$$p^{\Lambda} = 2Re\left[i\overline{Z}L^{\Lambda} + ig^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{L}^{\Lambda}\right] = -2Im\left[\overline{Z}L^{\Lambda} + g^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{L}^{\Lambda}\right]. \quad (134)$$

On the other hand, by using the second of Ansätze (42), the contraction of both sides of (130) with $g^{i\overline{j}}D_jM_{\Sigma}$ analogously yields

$$\begin{aligned} g^{i\overline{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\overline{j}}\overline{Z} &= q_{\Lambda}g^{i\overline{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\overline{j}}\overline{L}^{\Lambda} - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}g^{i\overline{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\overline{j}}\overline{L}^{\Delta} \\ &= q_{\Lambda}g^{i\overline{j}}\overline{\mathcal{N}}_{\Sigma\Delta}\left(D_iL^{\Delta}\right)\overline{D}_{\overline{j}}\overline{L}^{\Lambda} - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}g^{i\overline{j}}\overline{\mathcal{N}}_{\Sigma\Xi}\left(D_iL^{\Xi}\right)\overline{D}_{\overline{j}}\overline{L}^{\Delta}. \end{aligned} \quad (135)$$

Once again, by exploiting the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse, recalling the first of the Ansätze (42), and using (132), (135) can be further elaborated as follows:

$$\begin{aligned} g^{i\overline{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\overline{j}}\overline{Z} = q_{\Lambda}\overline{\mathcal{N}}_{\Sigma\Delta}&\left[-\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Delta\Lambda} - \overline{L}^{\Delta}L^{\Lambda}\right] - p^{\Lambda}\mathcal{N}_{\Lambda\Delta}\overline{\mathcal{N}}_{\Sigma\Xi} \\ &\times\left[-\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Xi\Delta} - \overline{L}^{\Xi}L^{\Delta}\right] = -\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Delta\Lambda}\left(Re\mathcal{N}_{\Sigma\Delta}\right)q_{\Lambda} \\ &+ \frac{i}{2}q_{\Sigma} - \overline{M}_{\Sigma}Z + \frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Xi\Delta}\left(Re\mathcal{N}_{\Sigma\Xi}\right)\left(Re\mathcal{N}_{\Lambda\Delta}\right)p^{\Lambda} \\ &+ \frac{1}{2}\left(Im\mathcal{N}_{\Lambda\Sigma}\right)p^{\Lambda}. \end{aligned} \quad (136)$$

Thence, by subtracting to the expression (136) its complex conjugate, one gets

$$q_{\Lambda} = 2Re\left[i\overline{Z}M_{\Lambda} + ig^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{M}_{\Lambda}\right] = -2Im\left[\overline{Z}M_{\Lambda} + g^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{M}_{\Lambda}\right]. \quad (137)$$

By expressing the identities (134) and (137) in a vector $Sp\left(2n_V+2\right)$-covariant notation, one finally gets

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix} = -2Im\left[\overline{Z}\begin{pmatrix} L^{\Lambda} \\ \\ M_{\Lambda} \end{pmatrix} + g^{i\overline{j}}D_iZ\begin{pmatrix} \overline{D}_{\overline{j}}\overline{L}^{\Lambda} \\ \\ \overline{D}_{\overline{j}}\overline{M}_{\Lambda} \end{pmatrix}\right], \quad (138)$$

or in compact form

$$Q^T = -2Im\left[\overline{Z}V + g^{i\overline{j}}\left(D_iZ\right)\overline{D}_{\overline{j}}\overline{V}\right], \quad (139)$$

where we recalled the definitions (1) and (25) of the $\left(2n_V+2\right)\times 1$ vectors Q^T and V, respectively.

It is worth pointing out that the vector identity (139) has been obtained only by using the properties of the SK geometry. The relations yielded by the identity (139) are $2n_V+2$ real ones, but they have been obtained by starting from an expression for

$\overline{D}_{\bar{i}}\overline{Z}$, corresponding to n_V complex, and therefore $2n_V$ real, degrees of freedom. The two redundant real degrees of freedom are encoded in the homogeneity (of degree 1) of the identity (139) under complex rescalings of the symplectic BH charge vector Q; indeed, by recalling the definition (23) it is immediate to check that the r.h.s. of identity (139) acquires an overall factor λ under a global rescaling of Q of the kind

$$Q \longrightarrow \lambda Q, \quad \lambda \in \mathbb{C}. \tag{140}$$

The summation of the expressions (133) and (136) with their complex conjugates, respectively, yields

$$(Im\mathscr{N})^{-1|\Delta\Lambda}(Re\mathscr{N}_{\Delta\Sigma})\,p^{\Sigma} - (Im\mathscr{N})^{-1|\Lambda\Sigma}q_{\Sigma} = 2Re\left[\overline{Z}L^{\Lambda} + g^{i\bar{j}}(D_iZ)\overline{D}_{\bar{j}}\overline{L}^{\Lambda}\right]; \tag{141}$$

$$\begin{aligned}&\left[Im\mathscr{N}_{\Lambda\Sigma} + (Im\mathscr{N})^{-1|\Xi\Delta}(Re\mathscr{N}_{\Lambda\Xi})Re\mathscr{N}_{\Sigma\Delta}\right]p^{\Sigma} - (Im\mathscr{N})^{-1|\Delta\Sigma}(Re\mathscr{N}_{\Lambda\Delta})\,q_{\Sigma}\\ &= 2Re\left[\overline{Z}M_{\Lambda} + g^{i\bar{j}}(D_iZ)\overline{D}_{\bar{j}}\overline{M}_{\Lambda}\right].\end{aligned} \tag{142}$$

In order to elaborate a shorthand notation for the obtained SKG identities (134), (137) and (141), (142), let us now reconsider the starting expressions (133) and (136), respectively, reading

$$\begin{aligned}&\left[\delta^{\Lambda}_{\Sigma} - i(Im\mathscr{N})^{-1|\Lambda\Delta}Re\mathscr{N}_{\Delta\Sigma}\right]p^{\Sigma} + i(Im\mathscr{N})^{-1|\Lambda\Sigma}q_{\Sigma}\\ &= -\,2i\overline{L}^{\Lambda}Z - 2ig^{i\bar{j}}\left(\overline{D}_{\bar{j}}\overline{Z}\right)D_iL^{\Lambda};\end{aligned} \tag{143}$$

$$\begin{aligned}&-\,i\left[(Im\mathscr{N})^{-1|\Xi\Delta}(Re\mathscr{N}_{\Lambda\Xi})Re\mathscr{N}_{\Sigma\Delta} + Im\mathscr{N}_{\Lambda\Sigma}\right]p^{\Sigma}\\ &+\left[\delta^{\Sigma}_{\Lambda} + i(Im\mathscr{N})^{-1|\Delta\Sigma}Re\mathscr{N}_{\Lambda\Delta}\right]q_{\Sigma} = -2i\overline{M}_{\Lambda}Z - 2ig^{i\bar{j}}\left(\overline{D}_{\bar{j}}\overline{Z}\right)D_iM_{\Lambda}.\end{aligned} \tag{144}$$

Thus, the identities (143) and (144) may be recast as the following fundamental $(2n_V+2)\times 1$ vector identity, defining the geometric structure of SK manifolds [9, 14, 17, 18, 23, 119]:

$$Q^T - i\epsilon\mathscr{M}(\mathscr{N})\,Q^T = -2i\overline{V}Z - 2ig^{i\bar{j}}\left(\overline{D}_{\bar{j}}\overline{Z}\right)D_iV. \tag{145}$$

The $(2n_V+2)\times(2n_V+2)$ real symmetric matrix $\mathscr{M}(\mathscr{N})$ is defined as [3, 4, 64]

$$\begin{aligned}&\mathscr{M}(\mathscr{N}) = \mathscr{M}(Re\mathscr{N}, Im\mathscr{N})\\ &\qquad\equiv \begin{pmatrix} Im\mathscr{N} + (Re\mathscr{N})(Im\mathscr{N})^{-1}Re\mathscr{N} & -(Re\mathscr{N})(Im\mathscr{N})^{-1}\\ -(Im\mathscr{N})^{-1}Re\mathscr{N} & (Im\mathscr{N})^{-1}\end{pmatrix},\end{aligned} \tag{146}$$

where $\mathscr{N}_{\Lambda\Sigma}$ is defined by (44). It is worth recalling that $\mathscr{M}(\mathscr{N})$ is symplectic with respect to the symplectic metric ϵ, i.e. it satisfies $((\mathscr{M}(\mathscr{N}))^T = \mathscr{M}(\mathscr{N}))$

$$\mathcal{M}\left(\mathcal{N}\right)\epsilon\mathcal{M}\left(\mathcal{N}\right)=\epsilon. \tag{147}$$

By using (28), (45), (46) and (47), the identity (145) implies the following relations:

$$\begin{cases} \left\langle V,Q^{T}-i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^{T}\right\rangle =-2Z; \\ \left\langle \overline{V},Q^{T}-i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^{T}\right\rangle =0; \\ \left\langle D_{i}V,Q^{T}-i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^{T}\right\rangle =0; \\ \left\langle \overline{D}_{\overline{i}}\overline{V},Q^{T}-i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^{T}\right\rangle =-2\overline{D}_{\overline{i}}\overline{Z}. \end{cases} \tag{148}$$

There are only $2n_V$ independent real relations out of the $4n_V+4$ real ones yielded by the $2n_V+2$ complex identities (145). Indeed, by taking the real and imaginary part of the SKG vector identity (145) one respectively obtains

$$Q^{T}=-2Re\left[iZ\overline{V}+ig^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_{i}V\right]=-2Im\left[\overline{Z}V+g^{i\overline{j}}\left(D_{i}Z\right)\overline{D}_{\overline{j}}\overline{V}\right]; \tag{149}$$

$$\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^{T}=2Im\left[iZ\overline{V}+ig^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_{i}V\right]=2Re\left[\overline{Z}V+g^{i\overline{j}}\left(D_{i}Z\right)\overline{D}_{\overline{j}}\overline{V}\right]. \tag{150}$$

Consequently, the imaginary and real parts of the SKG vector identity (145) are *linearly dependent* one from the other, being related by the $(2n_V+2)\times(2n_V+2)$ real matrix

$$\epsilon\mathcal{M}\left(\mathcal{N}\right)=\begin{pmatrix} \left(Im\mathcal{N}\right)^{-1}Re\mathcal{N} & -\left(Im\mathcal{N}\right)^{-1} \\ Im\mathcal{N}+\left(Re\mathcal{N}\right)\left(Im\mathcal{N}\right)^{-1}Re\mathcal{N} & -\left(Re\mathcal{N}\right)\left(Im\mathcal{N}\right)^{-1} \end{pmatrix}. \tag{151}$$

Put another way, (149) and (150) yield

$$Re\left[Z\overline{V}+g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_{i}V\right]=\epsilon\mathcal{M}\left(\mathcal{N}\right)Im\left[Z\overline{V}+g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_{i}V\right], \tag{152}$$

expressing the fact that the real and imaginary parts of the quantity $Z\overline{V}+g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_{i}V$ are simply related through a finite *symplectic rotation* given by the matrix $\epsilon\mathcal{M}\left(\mathcal{N}\right)$ (see (168) further below), whose simplecticity directly follows from the symplectic nature of $\mathcal{M}\left(\mathcal{N}\right)$. Equation (152) reduces the number of independent real relations implied by the identity (145) from $4n_V+4$ to $2n_V+2$. Two additional real degrees of freedom are scaled out by the complex rescaling (140).

This is clearly consistent with the fact that the $2n_V+2$ complex identities (145) express nothing but a *change of basis* of the BH charge configurations, between the Kähler-invariant $1\times(2n_V+2)$ symplectic (magnetic/electric) basis vector Q defined by (1) and the complex, moduli-dependent $1\times(n_V+1)$ *matter charges* vector (with Kähler weights $(1,-1)$)

$$\mathcal{Z}\left(z,\overline{z}\right)\equiv\left(Z\left(z,\overline{z}\right),Z_{i}\left(z,\overline{z}\right)\right)_{i=1,\ldots,n_V}. \tag{153}$$

It should be recalled that the BH charges are conserved due to the overall $(U(1))^{n_V+1}$ gauge-invariance of the system under consideration, and Q and $\mathscr{Z}(z,\bar{z})$ are two *equivalent* basis for them. Their very equivalence relations are given by the SKG identities (145) themselves. By its very definition (1), Q is *moduli-independent* (at least in a stationary, spherically symmetric and asymptotically flat extremal BH background, as is the case being treated here), whereas Z is *moduli-dependent*, since it refers to the eigenstates of the $\mathcal{N}=2,\ d=4$ supergravity multiplet and of the n_V Maxwell vector supermultiplets.

3.3.2 "New Attractor" Approach

The evaluation of the (real part of the) fundamental SK geometrical identities (138) and (139) along the constraints determining the various classes of critical points of V_{BH} in $\mathscr{M}_{n_V}$ allows one to obtain a completely equivalent form of the AEs for extremal (static, spherically symmetric, asymptotically flat) BHs in $\mathcal{N}=2,\ d=4$ ungauged supergravity, which may be simpler in some cases (see also [26] for the treatment of an explicit case).

(I) *Supersymmetric* ($\frac{1}{2}$*-BPS*) *critical points*. By evaluating the identities (138) and (139) along the constraints (56), one obtains

$$\begin{pmatrix} p^\Lambda \\ \\ q_\Lambda \end{pmatrix} = -2Im\left[e^{K/2}\overline{Z}\begin{pmatrix} X^\Lambda \\ \\ F_\Lambda \end{pmatrix}\right]_{\frac{1}{2}-BPS}, \tag{154}$$

or in compact form

$$Q^T = -2Im\left[e^{K/2}\overline{Z}\Pi\right]_{\frac{1}{2}-BPS}.$$

Equations (154) and (155) are equivalent, purely algebraic forms of the $\frac{1}{2}$-BPS extremal BH AEs, given by the (partly differential) conditions (56). By inserting as input the BH charge configuration $Q \equiv \left(p^\Lambda,\ q_\Lambda\right)$ and the covariantly holomorphic sections L^Λ and M_Λ of the $U(1)$-bundle over $\mathscr{M}_{n_V}$, (154) and (155) give as output (if any) the purely charge-dependent $\frac{1}{2}$-BPS critical points $\left(z^i_{\frac{1}{2}\text{-}BPS}(p,q),\ \bar{z}^{\bar{i}}_{\frac{1}{2}\text{-}BPS}(p,q)\right)$ of V_{BH}.

By looking at (154) and (155), it is easy to realize that $\frac{1}{2}$-BPS critical points of V_{BH} with $Z=0$ (which are *degenerate*, yielding $V_{BH,\frac{1}{2}\text{-}BPS}=0$) correspond to the trivial case of *all* vanishing magnetic and electric BH charges. This means that (static, spherically symmetric, asymptotically flat) extremal BHs with $\frac{1}{2}$-BPS attractor horizon scalar configurations with $Z=0$ (i.e. with no central extension of the $\mathcal{N}=2,\ d=4$ horizon supersymmetry algebra) cannot be described by the classical extremal BH Attractor Mechanism encoded by (154) and (155). They are a particular case of the so-called *"small"* extremal

BHs, which are *classically degenerate*, acquiring a non-vanishing, finite horizon area and entropy only taking into account quantum/higher-derivative corrections.

It is worth pointing out that (154) and (155) are purely algebraic ones, whereas (56) are (partly) differential, and thus, in general, more complicated to be solved. Consequently, at least in the $\frac{1}{2}$-BPS case, the *"new attractor" approach* is simpler of the *"criticality conditions" approach* to the search of critical points of V_{BH}.

(II) *Non-BPS $Z \neq 0$ critical points*. By evaluating the identities (138) and (139) along the constraints (59) and (60), one obtains

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix} = 2Im\left\{ e^{K/2}\left[Z\begin{pmatrix} \overline{X}^{\Lambda} \\ \\ \overline{F}_{\Lambda} \end{pmatrix} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}g^{i\overline{i}}g^{j\overline{j}}g^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)\begin{pmatrix} D_iX^{\Lambda} \\ \\ D_iF_{\Lambda} \end{pmatrix}\right]\right\}_{non-BPS,Z\neq 0}, \tag{155}$$

or in compact form

$$Q^T = 2Im\left\{ e^{K/2}\left[Z\overline{\Pi} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}g^{i\overline{i}}g^{j\overline{j}}g^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)D_i\Pi\right]\right\}_{non-BPS,Z\neq 0}. \tag{156}$$

Equations (155) and (156) are equivalent forms of the non-BPS $Z \neq 0$ extremal BH AEs, given by the (partly differential) conditions (59) and (60). By inserting as input the BH charge configuration Q, the covariantly holomorphic sections L^{Λ} and M_{Λ}, the Kähler potential K (and consequently the contravariant metric tensor $g^{i\overline{j}}$) and the completely symmetric, covariantly holomorphic rank-3 tensor C_{ijk}, (155) and (156) give as output (if any) the purely charge-dependent non-BPS $Z \neq 0$ critical points $\left(z^i_{non\text{-}BPS,Z\neq 0}(p,\, q),\, \overline{z}^{\overline{i}}_{non\text{-}BPS,Z\neq 0}(p,\, q)\right)$ of V_{BH}. Notice that, different from (154) and (155), (155) and (156) are not purely algebraic. Thus, in the non-BPS $Z \neq 0$ case the (computational) simplification in the search of critical points of V_{BH} obtained by exploiting the *"new attractor" approach* rather than the *"criticality conditions" approach* is model dependent.

It is interesting to point out that, as is evident by looking for instance at (156), at the non-BPS $Z \neq 0$ critical points of V_{BH} the coefficients of $\overline{\Pi}$ and $D_i\Pi$ in the AEs have the same holomorphicity in the central charge Z, i.e. they can be expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\overline{i}}\overline{Z}$. Such a fact does not happen in a generic point of $\mathcal{M}_{n_V}$, as is seen from the global identity (139). As is evident, the price to be paid in order to obtain the same holomorphicity in Z at the non-BPS $Z \neq 0$ critical points of V_{BH} is the fact that the coefficient of $D_i\Pi$ is not linear in some covariant derivative of Z any more, also explicitly depending on the rank-3 covariantly antiholomorphic tensor $\overline{C}_{\overline{i}\overline{j}\overline{k}}$.

(III) *Non-BPS* $Z=0$ *critical points*. By evaluating the identities (138) and (139) along the constraints (87) and (88), one obtains

$$\begin{pmatrix} p^\Lambda \\ \\ q_\Lambda \end{pmatrix} = 2Im\left\{ e^{K/2}\left[g^{i\overline{j}}\left(\overline{\partial}_{\overline{j}}\overline{Z}\right)\begin{pmatrix} D_iX^\Lambda \\ \\ D_iF_\Lambda \end{pmatrix}\right]\right\}_{non-BPS,Z=0}, \qquad (157)$$

or in compact form

$$Q^T = 2Im\left\{ e^{K/2}\left[g^{i\overline{j}}\left(\overline{\partial}_{\overline{j}}\overline{Z}\right) D_i\Pi\right]\right\}_{non-BPS,Z=0}. \qquad (158)$$

Equations (157) and (158) are equivalent forms of the non-BPS $Z=0$ extremal BH AEs given by the (partly differential) conditions (87) and (88). By inserting as input the BH charge configuration Q, the covariantly holomorphic sections L^Λ and M_Λ, and the Kähler potential K (and consequently the contravariant metric tensor $g^{i\overline{j}}$), Equations (157) and (158) give as output (if any) the purely charge-dependent non-BPS $Z=0$ critical points $\left(z^i_{non\text{-}BPS,Z=0}(p,q),\ \overline{z}^{\overline{i}}_{non\text{-}BPS,Z=0}(p,q)\right)$ of V_{BH}. Different from (154) and (155), and similar to (155) and (156), Equations (157) and (158) are not purely algebraic. Thus, in the non-BPS $Z=0$ case the (computational) simplification in the search of critical points of V_{BH} obtained by exploiting the *"new attractor" approach* rather than the *"criticality conditions" approach* is model dependent.

3.4 Type IIB Superstrings on CY$_3$

3.4.1 Hodge Decomposition of $\mathscr{H}_3$

We consider Type IIB superstring theory compactified on a Calabi-Yau threefold (CY_3) [64, 66, 67, 68, 120, 121], determining an effective $\mathscr{N}=2$, $d=4$ ungauged supergravity with a number n_V of Abelian vector multiplets. Within such a framework, the CY_3 has a complex structure (CS) moduli space (of complex dimension $n_V = h_{2,1} \equiv dim\left(H^{2,1}(CY_3)\right)$, where $H^{2,1}$ is the (2,1)-cohomology group of the considered manifold), which is a special Kähler (SK) manifold).

We introduce a[10] b_3-dim. real (manifestly symplectic-covariant) basis of the third real[11] cohomology $H^3(CY_3,\mathbb{R})$, given by the set of real 3-forms $\left\{\alpha_\Lambda,\ \beta^\Lambda\right\}$

[10] $b_3 = 2h_{2,1}+2$ is the so-called third Betti number of the CY_3.

[11] In the strict quantum regime, one should consider the third *integer* cohomology $H^3(CY_3,\mathbb{Z})$. The present (semi)classical treatment deal with the *large charges limit* and thus consistently consider real, unquantized, rather than integer, quantized quantities.

($\Lambda = 0, 1, \ldots, h_{2,1}$ throughout) satisfying[12]

$$\int_{CY_3} \alpha_\Lambda \wedge \alpha_\Sigma = 0, \quad \int_{CY_3} \beta^\Lambda \wedge \beta^\Sigma = 0, \quad \int_{CY_3} \alpha_\Lambda \wedge \beta^\Sigma = \delta_\Lambda^\Sigma. \tag{159}$$

By Poincarè-duality on CY_3, we may correspondingly introduce the b_3-dim. real (manifestly symplectic-covariant) basis of the third real homology $H_3(CY_3, \mathbb{R})$, given by the set of real 3-cycles $\{A^\Lambda, B_\Lambda\}$ satisfying

$$\int_{A^\Lambda} \alpha_\Sigma = \delta_\Sigma^\Lambda, \quad \int_{A^\Lambda} \beta^\Sigma = 0, \quad \int_{B_\Lambda} \alpha_\Sigma = 0, \int_{B_\Lambda} \beta^\Sigma = -\delta_\Lambda^\Sigma. \tag{160}$$

The CY_3 is endowed with a (*nowhere-vanishing*) holomorphic 3-form

$$\Omega_3(z) \equiv X^\Lambda(z)\,\alpha_\Lambda - F_\Lambda(z)\,\beta^\Lambda \in H^{3,0}(CY_3), \tag{161}$$

where "z" denotes the functional dependence on the CS moduli $\left\{z^i, \bar{z}^{\bar{i}}\right\}$ ($i = 1, \ldots, h_{2,1}$ throughout), and $\{X^\Lambda, F_\Lambda\}$ stands for the basis of symplectic holomorphic fundamental periods of Ω_3 around the 3-cycles $\{A^\Lambda, B_\Lambda\}$, respectively:

$$X^\Lambda(z) \equiv \int_{A^\Lambda} \Omega_3(z), \quad F_\Lambda(z) \equiv \int_{B_\Lambda} \Omega_3(z). \tag{162}$$

Ω_3, as well as its fundamental periods, has Kähler weights (2,0):

$$\begin{aligned} D_i\Omega_3 &= \partial_i\Omega_3 + (\partial_i K)\,\Omega_3, \\ \overline{D}_{\bar{i}}\Omega_3 &= \overline{\partial}_{\bar{i}}\Omega_3 = 0, \end{aligned} \tag{163}$$

where K is the real Kähler potential in the $h_{2,1}$-dim. SK CS moduli space of CY_3.

Type IIB compactified on CY_3 is characterized by a real 5-form

$$\mathscr{Z} \equiv \mathscr{F}^\Lambda \alpha_\Lambda - \mathscr{G}_\Lambda \beta^\Lambda, \tag{164}$$

where $\mathscr{F}^\Lambda$ is the space-time 2-form given by the Abelian field-strengths ($\Lambda = 0$ pertains to the graviphoton, whereas $\Lambda = i$ corresponds to the Maxwell vector supermultiplets), and $\mathscr{G}_\Lambda$ is the corresponding "dual" space-time 2-form, in the sense of Legendre transform:

$$\mathscr{G}_\Lambda \equiv \frac{\delta \mathscr{L}}{\delta \mathscr{F}^\Lambda} = (ReN_{\Lambda\Sigma})\,\mathscr{F}^\Sigma + \frac{1}{2}\,(ImN_{\Lambda\Sigma})^*\,\mathscr{F}^\Sigma. \tag{165}$$

${}^*\mathscr{F}^\Sigma$ denotes the Hodge $*$-dual of $\mathscr{F}^\Lambda$, defined in components as follows (the space-time indices μ, ν run 0,1,2,3 throughout):

[12] Recall that the $\wedge$ ("wedge") product among odd-forms is odd, whereas the one among even-forms (and among odd- and even-forms) is even.

$$^*\mathscr{F}^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathscr{F}^{\Lambda|\rho\sigma} = \frac{1}{2}G^{\rho\lambda}G^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathscr{F}^{\Lambda}_{\lambda\tau}, \tag{166}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the $d=4$ completely antisymmetric Ricci-Levi-Civita tensor, and $G^{\mu\nu}$ is the $d=4$ space-time completely contravariant metric tensor. $\mathscr{L}$ stands for the (bosonic sector of the) $\mathscr{N}=2,\ d=4$ ungauged supergravity Lagrangian density:

$$\begin{aligned}\mathscr{L} = &-\frac{R}{2} + g_{i\bar{j}}\left(\partial_\mu z^i\right)\left(\partial_\nu \bar{z}^{\bar{j}}\right)G^{\mu\nu} + \frac{1}{4}\left(ImN_{\Lambda\Sigma}\right)G^{\mu\lambda}G^{\nu\rho}\mathscr{F}^{\Lambda}_{\mu\nu}\mathscr{F}^{\Lambda}_{\lambda\rho}\\ &+\frac{1}{4}\left(ReN_{\Lambda\Sigma}\right)G^{\mu\lambda}G^{\nu\rho}\mathscr{F}^{\Lambda}_{\mu\nu}{}^*\mathscr{F}^{\Lambda}_{\lambda\rho}.\end{aligned} \tag{167}$$

The Hodge $*$-duality acts as a symplectic $Sp\left(2h_{1,2}+2,\mathbb{R}\right)$ rotation on the basis $\left\{\alpha_\Lambda,\ \beta^\Lambda\right\}$:

$$\begin{pmatrix} ^*\alpha_\Lambda \\ ^*\beta^\Lambda \end{pmatrix} = \mathscr{S}\begin{pmatrix} \alpha_\Lambda \\ \beta^\Lambda \end{pmatrix},\quad \mathscr{S} \equiv -\epsilon\mathscr{M}\left(\mathscr{N}\right),\ \mathscr{S}^T\epsilon\mathscr{S} = \epsilon. \tag{168}$$

where ϵ is the $(2h_{1,2}+2)$-dim. symplectic metric defined in (24), $\mathscr{M}\left(\mathscr{N}\right)$ is the real, symplectic matrix defined by (146). Notice that $\mathscr{S}$ is nothing but the opposite of the matrix given by (151). It can be shown that $\mathscr{Z}$ is Hodge $*$-self-dual:

$$^*\mathscr{Z} = \mathscr{Z}. \tag{169}$$

Whenever the relevant integrations over internal manifold CY_3 and over space-time make sense, manifestly symplectic-covariant magnetic and electric charges can be introduced as the asymptotical "space-dressings" of a suitable contraction of $\mathscr{Z}$ over the symplectic 3-cycles of CY_3, i.e. as the asymptotical fluxes of the space-time 2-forms corresponding to the components of $\mathscr{Z}$ along the symplectic basis $\left\{A^\Lambda,\ B_\Lambda\right\}$ of $H_3\left(CY_3,\mathbb{R}\right)$, respectively:

$$\begin{aligned} p^\Lambda &\equiv \frac{1}{4\pi}\int_{A^\Lambda\times S^2_\infty}\mathscr{Z} = \frac{1}{4\pi}\int_{A^\Lambda\times S^2_\infty}\left(\mathscr{F}^\Sigma\alpha_\Sigma - \mathscr{G}_\Sigma\beta^\Sigma\right) = \frac{1}{4\pi}\int_{S^2_\infty}\mathscr{F}^\Lambda;\\ q_\Lambda &\equiv \frac{1}{4\pi}\int_{B_\Lambda\times S^2_\infty}\mathscr{Z} = \frac{1}{4\pi}\int_{B_\Lambda\times S^2_\infty}\left(\mathscr{F}^\Sigma\alpha_\Sigma - \mathscr{G}_\Sigma\beta^\Sigma\right) = \frac{1}{4\pi}\int_{S^2_\infty}\mathscr{G}^\Lambda, \end{aligned} \tag{170}$$

where S^2_∞ denotes the 2-sphere at spatial infinity[13].

$\left\{p^\Lambda,\ q_\Lambda\right\}$ can be seen as the components (along the real symplectic basis $\left\{\alpha_\Lambda,\ \beta^\Lambda\right\}$ of $H^3\left(CY_3,\mathbb{R}\right)$) of the real flux 3-form $\mathscr{H}_3$, defined as the asymptotical "space-dressing" of $\mathscr{Z}$:

$$\mathscr{H}_3 \equiv \frac{1}{4\pi}\int_{S^2_\infty}\mathscr{Z} = p^\Lambda\alpha_\Lambda - q_\Lambda\beta^\Lambda \in H^3\left(CY_3,\mathbb{R}\right). \tag{171}$$

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[13] Consistently with (a proper subset of) the solutions of $\mathscr{N}=2,\ d=4$ ungauged supergravity, the space-time metric is assumed to be static, spherically symmetric, and asymptotically flat. In such a framework, "spatial infinity" corresponds to $r\longrightarrow\infty$, where r is the radial coordinate.

$\{p^\Lambda,\ q_\Lambda\}$ are the physical charges, and they are conserved, due to the overall $(U(1))^{h_{2,1}+1}$ gauge symmetry of the considered framework. They, respectively, are the magnetic and electric charges of the $(U(1))^{h_{2,1}+1}$ gauge group of the (symplectic) real parameterization of $H^3(CY_3,\ \mathbb{R})$, which however is not the only possible one.

Indeed, in general the third real cohomology $H^3(CY_3,\mathbb{R})$ can be Hodge-decomposed along the third Dalbeault cohomogy of CY_3 as follows:

$$H^3(CY_3,\mathbb{R}) = H^{3,0}(CY_3) \oplus_s H^{2,1}(CY_3) \oplus_s H^{1,2}(CY_3) \oplus_s H^{0,3}(CY_3), \qquad (172)$$

corresponding to perform a change of basis from the symplectic real basis to the Dalbeault basis:

$$\left\{\alpha_\Lambda, \beta^\Lambda\right\} \longrightarrow \left\{\Omega_3, D_i\Omega_3, \overline{D}_{\overline{i}}\overline{\Omega}_3, \overline{\Omega}_3\right\}. \qquad (173)$$

The subscript "s" in in (172) stands for the semidirect cohomological sum, due to the fact that (some of the) cohomologies in the r.h.s. of the Hodge decomposition (172) have non-vanishing intersections. Indeed, as it can be checked by recalling (45), (46) and (159), the following results hold:

$$\begin{aligned}
&\int_{CY_3} \Omega_3 \wedge \Omega_3 = 0,\ \int_{CY_3} \Omega_3 \wedge D_i\Omega_3 = 0,\ \int_{CY_3} \Omega_3 \wedge \overline{D}_{\overline{i}}\overline{\Omega}_3 = 0,\\
&\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3 = -ie^{-K} \Leftrightarrow K = -ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right),\\
&\int_{CY_3} (D_i\Omega_3) \wedge D_j\Omega_3 = 0,\\
&\int_{CY_3} (D_i\Omega_3) \wedge \overline{D}_{\overline{j}}\overline{\Omega}_3 = \left[\overline{\partial}_{\overline{j}}\partial_i ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right)\right]\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\\
&\qquad = -\overline{\partial}_{\overline{j}}\partial_i K \int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3 = ie^{-K} g_{i\overline{j}}\\
&\qquad\qquad \Updownarrow\\
&g_{i\overline{j}} = -\overline{\partial}_{\overline{j}}\partial_i ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right) = -\frac{\int_{CY_3} (D_i\Omega_3) \wedge \overline{D}_{\overline{j}}\overline{\Omega}_3}{\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3}.
\end{aligned} \qquad (174)$$

In particular, the second line of (174) allows one to write the covariant derivatives of Ω_3 (which are the basis of $H^{2,1}(CY_3)$) as follows:

$$D_i\Omega_3 = \left(\partial_i - \frac{\int_{CY_3} (\partial_i\Omega_3) \wedge \overline{\Omega}_3}{\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3}\right)\Omega_3. \qquad (175)$$

It is worth pointing out that the $2h_{2,1}+2$ 3-forms $\left\{\Omega_3, D_i\Omega_3, \overline{D}_{\overline{i}}\overline{\Omega}_3, \overline{\Omega}_3\right\}_{i=1,\dots,h_{2,1}}$ are *all* the possible ((2,0) and (0,2))-Kähler-weighted independent 3-forms which can be defined on CY_3 in the considered framework. This is due to the two fundamental relations

$$\overline{D}_{\overline{j}} D_i \Omega_3 = g_{i\overline{j}} \Omega_3; \tag{176}$$

$$D_i D_j \Omega_3 = i C_{ijk} g^{k\overline{l}} \overline{D}_{\overline{l}} \overline{\Omega}_3 = D_{(i} D_{j)} \Omega_3, \tag{177}$$

which are the translation, in the language of forms on CY_3, of the third and second of (38), respectively. Notice that the third of (38) and (176) hold in a generic Kähler framework, whereas the second of (38) and (177) in general hold only in SK geometry. Due to (177), the completely symmetric, covariantly holomorphic tensor C_{ijk} of SK geometry can be obtained by intersecting the elements of the basis of $H^{2,1}(CY_3)$ with their covariant derivatives (and normalizing with respect to the intersection of $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$):

$$C_{ijk} = C_{(ijk)} = -i \frac{\int_{CY_3} (D_i D_j \Omega_3) \wedge D_k \Omega_3}{\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3} = e^K \int_{CY_3} (D_i D_j \Omega_3) \wedge D_k \Omega_3. \tag{178}$$

According to the Hodge-decomposition (172) implemented through the change of basis (173), the charges undergo the following change of basis:

$$\left\{ p^\Lambda, q_\Lambda \right\} \longrightarrow \left\{ Z^{3,0}(z,p,q), Z_i^{2,1}(z,\overline{z},p,q), Z_{\overline{i}}^{1,2}(z,\overline{z},p,q), Z^{0,3}(\overline{z},p,q) \right\}, \tag{179}$$

where the complex, (CS) moduli-dependent quantities on the r.h.s. are defined as follows:

$$\begin{aligned} Z^{3,0}(z;p,q) &\equiv \int_{CY_3} \mathscr{H}_3 \wedge \Omega_3(z) = \frac{1}{4\pi} \int_{CY_3} \left(\int_{S^2_\infty} \mathscr{L} \right) \wedge \Omega_3(z) \\ &= X^\Lambda(z) q_\Lambda - F_\Lambda(z) p^\Lambda = W(z;p,q); \end{aligned} \tag{180}$$

$$\begin{aligned} Z_i^{2,1}(z,\overline{z};p,q) &\equiv \int_{CY_3} \mathscr{H}_3 \wedge (D_i \Omega_3)(z,\overline{z}) = \int_{CY_3} \left(\int_{S^2_\infty} \mathscr{L} \right) \wedge (D_i \Omega_3)(z,\overline{z}) \\ &= \left(D_i X^\Lambda \right)(z,\overline{z}) q_\Lambda - (D_i F_\Lambda)(z,\overline{z}) p^\Lambda = (D_i W)(z,\overline{z};p,q); \end{aligned} \tag{181}$$

$$Z_{\overline{i}}^{1,2}(z,\overline{z};p,q) \equiv \overline{Z_i^{2,1}(z,\overline{z};p,q)}; \tag{182}$$

$$Z^{0,3}(\overline{z};p,q) \equiv \overline{Z^{3,0}(z;p,q)}. \tag{183}$$

As can be seen, $Z^{3,0}(z;p,q)$ and $Z_i^{2,1}(z,\overline{z};p,q)$ are, respectively, nothing but the $\mathscr{N}=2,\ d=4$ holomorphic central charge function $W(z;p,q)$, also named $\mathscr{N}=2$ superpotential (see (23) and comments below), and its covariant derivatives, introduced *à la Gukov-Vafa-Witten* (GVW) [90, 91] also in the considered $\mathscr{N}=2,\ d=4$ framework. In other words, (180) defines the holomorphic central extension of the $\mathscr{N}=2,\ d=4$ local supersymmetry algebra, whereas (180) defines in a geometrical way the charges of the other field strength vectors, orthogonal to the graviphoton. $\left\{Z^{3,0}, Z_i^{2,1}, Z_{\bar{i}}^{1,2}, Z^{0,3}\right\}$ correspond to electric and magnetic charges of the $(U(1))^{h_{2,1}+1}$ gauge group of the complex Dalbeault parameterization of $H^3(CY_3, \mathbb{R})$. Their dependence on moduli can be understood by taking into account that they refer to the supermultiplet eigenstates, which are moduli-dependent (as already pointed out below (152)). They satisfy the following model-independent sum rules [3]:

$$\begin{aligned}\left(\left|Z^{3,0}\right|^2 + g^{i\bar{j}} Z_i^{2,1} Z_{\bar{j}}^{1,2}\right) e^K &= -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right) \mathscr{M}(\mathscr{N}) \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} = I_1(z, \bar{z}; p, q) \\ &= V_{BH}(z, \bar{z}; p, q) \geqslant 0;\end{aligned} \tag{184}$$

$$\left(\left|Z^{3,0}\right|^2 - g^{i\bar{j}} Z_i^{2,1} Z_{\bar{j}}^{1,2}\right) e^K = -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right) \mathscr{M}(\mathscr{F}) \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} = I_2(z, \bar{z}; p, q) \gtreqless 0, \tag{185}$$

where $\mathscr{M}(\mathscr{N})$ is the real, symplectic matrix defined by (146), $\mathscr{F} \equiv F_{\Lambda\Sigma} = \partial_\Sigma F_\Lambda$, $\mathscr{M}(\mathscr{F}) = \mathscr{M}(\mathscr{N} \to \mathscr{F})$. I_1 and I_2 are the first and second lowest-order (quadratic in charges) invariants of SK geometry. As far as the metric $g_{i\bar{j}}$ of the SK CS moduli space is regular, I_1 has positive signature and it is nothing but the "BH effective potential" V_{BH}, whereas I_2 has signature $(1, h_{2,1})$. Since the considered extremal BH background is static (and spherically symmetric), the undressed charges p^Λ and q_Λ are conserved in time, and so are the dressed charges $\left\{Z^{3,0}, Z_i^{2,1}, Z_{\bar{i}}^{1,2}, Z^{0,3}\right\}$ (which however, through their dependence on scalars, do depend on radial coordinate).

The real, Kähler gauge-invariant 3-form $\mathscr{H}_3$ can be thus Hodge-decomposed as follows $(\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{C})$

$$\begin{aligned}\mathscr{H}_3 &= e^K \left[\begin{array}{l} \gamma_1 \left(\int_{CY_3} \mathscr{H}_3 \wedge \Omega_3\right) \overline{\Omega}_3 + \gamma_2 g^{i\bar{j}} \left(\int_{CY_3} \mathscr{H}_3 \wedge (D_i\Omega_3)\right) \overline{D}_{\bar{j}}\overline{\Omega}_3 + \\ + \gamma_3 g^{j\bar{i}} \left(\int_{CY_3} \mathscr{H}_3 \wedge \overline{D}_{\bar{i}}\overline{\Omega}_3\right) D_j\Omega_3 + \gamma_4 \left(\int_{CY_3} \mathscr{H}_3 \wedge \overline{\Omega}_3\right) \Omega_3 \end{array}\right] \\ &= e^K \left[\begin{array}{l} \gamma_1 W \overline{\Omega}_3 + \gamma_2 g^{i\bar{j}} (D_i W) \overline{D}_{\bar{j}}\overline{\Omega}_3 + \\ + \gamma_3 g^{j\bar{i}} \left(\overline{D}_{\bar{i}}\overline{W}\right) D_j\Omega_3 + \gamma_4 \overline{W}\Omega_3 \end{array}\right],\end{aligned} \tag{186}$$

where (180), (181), (182), (183) were used. The r.h.s. of the Hodge-decomposition (186) is the most general Kähler gauge-invariant combination of *all* the possible ((2,0) and (0,2))-Kähler-weighted independent 3-forms for Type IIB on CY_3s. The overall factor e^K (with Kähler weights $(-2,-2)$) is necessary to make the r.h.s. of the identity (186) Kähler gauge-invariant. The reality condition $\overline{\mathscr{H}}_3 = \mathscr{H}_3$ implies $\gamma_3 = \overline{\gamma_2}$ and $\gamma_4 = \overline{\gamma_1}$. The complex coefficients γ_1 and γ_2 can be determined by computing $\int_{CY_3} \mathscr{H}_3 \wedge \Omega_3$ and $\int_{CY_3} \mathscr{H}_3 \wedge D_l\Omega_3$, using the identity (186) and recalling (180), (181), (182), (183) and the intersections (174). By doing so, one obtains:

$$\begin{aligned} W &= \int_{CY_3} \mathscr{H}_3 \wedge \Omega_3 = e^K \gamma_1 W \int_{CY_3} \overline{\Omega}_3 \wedge \Omega_3 = i\gamma_1 W \Leftrightarrow \gamma_1 = -i; \\ D_l W &= \int_{CY_3} \mathscr{H}_3 \wedge D_l\Omega_3 = e^K \gamma_2 g^{i\bar{j}} (D_i W) \int_{CY_3} \overline{D}_{\bar{j}}\overline{\Omega}_3 \wedge D_l\Omega_3 = -i\gamma_2 D_l W \Leftrightarrow \gamma_2 = i. \end{aligned} \tag{187}$$

Thus, the complete Hodge-decomposition of the real flux 3-form $\mathscr{H}_3$ of Type IIB on CY_3s reads

$$\begin{aligned} \mathscr{H}_3 &= -ie^K \left[W\overline{\Omega}_3 - g^{i\bar{j}} (D_i W) \overline{D}_{\bar{j}}\overline{\Omega}_3 + g^{j\bar{i}} \left(\overline{D}_{\bar{i}}\overline{W}\right) D_j\Omega_3 - \overline{W}\Omega_3 \right] \\ &= -2Im \left[\overline{Z}\hat{\Omega}_3 - g^{j\bar{i}} \left(\overline{D}_{\bar{i}}\overline{Z}\right) D_j\hat{\Omega}_3 \right], \end{aligned} \tag{188}$$

where in the second line we recalled the definition of the $\mathscr{N} = 2,\ d = 4$ covariantly holomorphic central charge function $Z(z,\bar{z},p,q)$ (with Kähler weights $(1,-1)$) given by (23) (see also (37)), and introduced the covariantly holomorphic (3,0)-form $\hat{\Omega}_3$ (with Kähler weights $(1,-1)$) on CY_3:

$$\begin{aligned} Z(z,\bar{z};p,q) &\equiv e^{\frac{K(z,\bar{z})}{2}} W(z;p,q), \\ D_i Z &= e^{\frac{K}{2}} D_i W,\ \overline{D}_{\bar{i}} Z = 0; \end{aligned} \tag{189}$$

$$\begin{aligned} \hat{\Omega}_3(z,\bar{z}) &\equiv \frac{\Omega_3}{\sqrt{i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3}} = e^{\frac{K(z,\bar{z})}{2}} \Omega_3(z), \\ D_i\hat{\Omega}_3 &= e^{\frac{K}{2}} D_i\Omega_3,\ \overline{D}_{\bar{i}}\hat{\Omega}_3 = 0. \end{aligned} \tag{190}$$

Let us now compare the Hodge-decomposition identity (188) with the real part (138) and (139) of the SK geometrical identities (145). It is immediate to realize that the identity (188) is nothing but the translation, in the language of forms of Type IIB on CY_3 (i.e. in a particular stringy framework) of the identity (138) and (139), which is the real part of the fundamental identities (145), holding for any SK geometry, irrespective of its microscopic/stringy origin.

3.4.2 "New Attractor" Approach

The evaluation of the Hodge-decomposition identity (188) along the constraints determining the various classes of critical points of V_{BH} in $\mathscr{M}_{n_V}$ (which in the considered stringy framework is nothing but the CS moduli space of CY_3) allows one to obtain a completely equivalent form of the AEs for extremal (static, spherically symmetric, asymptotically flat) BHs in the particular framework in which $\mathcal{N}=2$, $d=4$ ungauged supergravity is obtained by compactifying Type IIB on CY_3. As already pointed out in the treatment at macroscopic level, in some cases such equivalent forms of AEs may be simpler to solve than the AEs obtained by exploiting the *"criticality conditions" approach* (see Sect. 3.1).

(I) *Supersymmetric ($\frac{1}{2}$-BPS) critical points*. By evaluating the Hodge-decomposition identity (188) along the constraints (56), one obtains

$$\begin{aligned} \mathscr{H}_3 &= -i\left[e^K\left(W\overline{\Omega}_3 - \overline{W}\Omega_3\right)\right]_{\frac{1}{2}-BPS} \\ &= -2Im\left(\overline{Z}\hat{\Omega}_3\right)_{\frac{1}{2}-BPS}. \end{aligned} \tag{191}$$

Equation (191) is the translation, for Type IIB on CY_3, of (154) and (155), which in turn are equivalent, purely algebraic forms of the $\frac{1}{2}$-BPS extremal BH AEs, given by the (partly differential) conditions (56). By recalling (172) and (173), (191) implies that at $\frac{1}{2}$-BPS critical points of V_{BH} the real flux 3-form $\mathscr{H}_3$ of Type IIB on CY_3 has vanishing components along the Dalbeault third cohomologies $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$. This can be understood easily by recalling (181):

$$D_iW = 0,\ \forall i \Longleftrightarrow \begin{cases} \int_{CY_3} \mathscr{H}_3 \wedge D_i\Omega_3 = 0, \forall i; \\ \Updownarrow \\ \int_{CY_3} \mathscr{H}_3 \wedge \overline{D}_{\overline{i}}\overline{\Omega}_3 = 0, \forall \overline{i}. \end{cases} \tag{192}$$

Thus, at $\frac{1}{2}$-BPS critical points of V_{BH} $\mathscr{H}_3$ is *"orthogonal"* (in the sense of (192), as understood below, as well) to *all* the 3-forms which are basis elements of $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$.

Consequently, the complete supersymmetry breaking at the horizon of (static, spherically symmetric, asymptotically flat) extremal BHs in $\mathcal{N}=2$, $d=4$ supergravity as low-energy, effective theory of Type IIB on CY_3 can be traced back to the non-vanishing *"intersections"* (defined by (181) and (182)) of $\mathscr{H}_3$ with $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$. Moreover, in light of (180), the $\frac{1}{2}$-BPS non-degeneracy condition $W_{\frac{1}{2}\text{-}BPS} \neq 0$ corresponds to a condition of non(-complete)- *"orthogonality"* between $\mathscr{H}_3$ and Ω_3, basis of $H^{3,0}(CY_3)$:

$$W \neq 0 \Longleftrightarrow \begin{cases} \int_{CY_3} \mathscr{H}_3 \wedge \Omega_3 \neq 0; \\ \Updownarrow \\ \int_{CY_3} \mathscr{H}_3 \wedge \overline{\Omega}_3 \neq 0. \end{cases} \tag{193}$$

(II) *Non-BPS $Z \neq 0$ critical points*. By evaluating the Hodge-decomposition identity (188) along the constraints (59) and (60), one obtains

$$\mathscr{H}_3 = 2Im\left[Z\overline{\hat{\Omega}}_3 + \frac{i}{2}\frac{Z}{|Z|^2}C_{ikl}g^{i\bar{j}}g^{k\bar{k}}g^{l\bar{l}}\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{j}}\overline{\hat{\Omega}}_3\right]_{non-BPS,Z\neq 0}, \tag{194}$$

Equation (194) is the translation, for Type IIB on CY_3, of (155) and (156), which in turn are equivalent forms of the non-BPS $Z \neq 0$ extremal BH AEs, given by the (partly differential) conditions (59) and (60).

By recalling (172) and (173), (194) implies that at the non-BPS $Z \neq 0$ critical points of V_{BH} the real flux 3-form $\mathscr{H}_3$ of type IIB on CY_3 has components along $H^{0,3}(CY_3)$ and $H^{2,1}(CY_3)$ with the same holomorphicity in the holomorphic central charge Z. In other words, such components can be expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\bar{i}}\overline{Z}$. Such a fact does not happen in a generic point of the CS moduli space of CY_3, as is seen from the global Hodge-decomposition identity (188). As is evident, the price to be paid in order to obtain the same holomorphicity in Z at the non-BPS $Z \neq 0$ critical points of V_{BH} is the fact that the component of $\mathscr{H}_3$ along $H^{2,1}(CY_3)$ is not linear in some covariant derivative of Z any more, also explicitly depending on the rank-3 covariantly antiholomorphic tensor $\overline{C}_{\bar{i}\bar{j}\bar{k}}$. By recalling (180), (181), (182), (183), this can be understood by considering the translation of (60) in the language of (3-)forms of Type IIB on CY_3:

$$\int_{CY_3}\mathscr{H}_3\wedge\overline{D}_{\bar{i}}\overline{\Omega}_3 = \frac{i}{2\int_{CY_3}\mathscr{H}_3\wedge\Omega_3}\overline{C}_{\bar{i}\bar{j}\bar{k}}g^{l\bar{j}}g^{m\bar{k}}\left(\int_{CY_3}\mathscr{H}_3\wedge D_l\Omega_3\right)$$
$$\times\int_{CY_3}\mathscr{H}_3\wedge D_m\Omega_3,\ \forall\bar{i} = \bar{1},\ldots,\overline{n_V}. \tag{195}$$

Equation (195), holding at non-BPS $Z \neq 0$ critical points of V_{BH}, expresses the *"intersections"* of $\mathscr{H}_3$ with $H^{1,2}(CY_3)$ (i.e. the components of $\mathscr{H}_3$ along $H^{2,1}(CY_3)$; see (186)) non-linearly in terms of *"intersections"* of $\mathscr{H}_3$ with $H^{3,0}(CY_3)$ and $H^{2,1}(CY_3)$, which can all be expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\bar{i}}\overline{Z}$.

(III) *Non-BPS* $Z = 0$ *critical points*. By evaluating the Hodge-decomposition identity (188) along the constraints (87) and (88), one obtains

$$\begin{aligned}\mathscr{H}_3 &= -ie^K\left[-g^{i\bar{j}}(\partial_iW)\overline{D}_{\bar{j}}\overline{\Omega}_3 + g^{j\bar{i}}\left(\overline{\partial}_{\bar{i}}\overline{W}\right)D_j\Omega_3\right]\\ &= 2Im\left[g^{j\bar{i}}\left(\overline{\partial}_{\bar{i}}\overline{Z}\right)D_j\hat{\Omega}_3\right],\end{aligned} \tag{196}$$

Equation (196) is the translation for Type IIB on CY_3 of (157) and (158), which in turn are equivalent forms of the non-BPS $Z = 0$ extremal BH AEs, given by the (partly differential) conditions (87) and (88). By recalling (172) and (173), (196) implies that at non-BPS $Z = 0$ critical points of V_{BH}, in an opposite fashion with respect to the case of $\frac{1}{2}$-BPS critical points of V_{BH}, the real flux 3-form $\mathscr{H}_3$ of Type IIB on CY_3 has vanishing components along the Dalbeault third cohomologies $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$. This can be understood easily by recalling (180):

$$W=0 \Longleftrightarrow \begin{cases} \int_{CY_3} \mathscr{H}_3 \wedge \Omega_3 = 0; \\ \Updownarrow \\ \int_{CY_3} \mathscr{H}_3 \wedge \overline{\Omega}_3 = 0. \end{cases} \tag{197}$$

Thus, at non-BPS $Z=0$ critical points of V_{BH} $\mathscr{H}_3$ is *"orthogonal"* (in the sense of (197), as understood below, as well) to Ω_3 and $\overline{\Omega}_3$, basis of $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$, respectively. Moreover, in light of (181) and (182), the non-BPS $Z=0$ non-degeneracy condition (*at least* for strictly positive-definite $g_{i\bar{j}}$ at the considered critical points of V_{BH})

$$(D_iW)_{non-BPS,Z=0} \neq 0, \textit{at least} \text{ for some } i \in \{1,\dots,n_V\} \tag{198}$$

corresponds to a condition of non(-complete)-*"orthogonality"* between $\mathscr{H}_3$ and the $D_i\Omega_3$s, basis elements of $H^{2,1}(CY_3)$:

$$\begin{gathered} D_iW \neq 0, \textit{at least} \text{ for some } i \in \{1,\dots,n_V\} \Longleftrightarrow \\ \begin{cases} \int_{CY_3} \mathscr{H}_3 \wedge D_i\Omega_3 \neq 0, , \textit{at least} \text{ for some } i \in \{1,\dots,n_V\}; \\ \Updownarrow \\ \int_{CY_3} \mathscr{H}_3 \wedge \overline{D}_{\bar{i}}\overline{\Omega}_3 \neq 0, \textit{at least} \text{ for some } \bar{i} \in \left\{\bar{1},\dots,\overline{n_V}\right\}. \end{cases} \end{gathered} \tag{199}$$

4 Flux Vacua Attractor Equations in $\mathscr{N}=1$, $d=4$ Supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$

4.1 CY$_3$ *Orientifolds*

We consider Type IIB superstring theory compactified on a CY_3 orientifold with O3/O7-planes (as in the GPK-KKLT model [87, 122]), determining an $\mathscr{N}=1$, $d=4$ supergravity as effective, low-energy theory. Within such a framework, we will derive FV AEs[14], similar to what is done in Sect. 3.4 for extremal BH AEs in effective $\mathscr{N}=2$, $d=4$ supergravity from Type IIB on CY_3. We will mainly follow [9, 92, 93]. In our treatment, the relevant moduli moduli space M of the CY_3 orientifold is the one composed by the (direct) product of the CS moduli space (of complex dimension $h_{2,1} \equiv dim\left(H^{2,1}(CY_3)\right)$, which is a SK manifold, and the 1-dim. Kähler manifold parameterized by the universal axion-dilaton. We will denote the

[14] In [123] the FV Attractor Mechanism has been shown to act also in the landscape of non-Kähler vacua emerging in the flux compactifications of heterotic superstrings.

CS moduli by $\left(x^i,\overline{x}^{\overline{i}}\right)_{i=1,\dots,h_{2,1}} \equiv \left(t^i,\overline{t}^{\overline{i}}\right)_{i=1,\dots,h_{2,1}}$ (not to be confused with the projective coordinates in the SK CS moduli space) and the axion-dilaton by $\tau \equiv t^0$:

$$M = \mathcal{M}_{t^0} \otimes \mathcal{M}_{CS}. \tag{200}$$

No Kähler structure (KS) moduli will be considered in our treatment of the classical FV Attractor Mechanism; indeed, in the considered framework the stabilization of KS moduli requires quantum perturbative or non-perturbative mechanisms, such as worldsheet instantons and gaugino condensation (see e.g. [87]).

4.1.1 Vielbein and Metric Tensor in the Moduli Space

We start by defining the structure of the $(h_{2,1}+1)$-dim. Kähler manifold spanned by the CS moduli and the axion-dilaton. Its Kähler potential can be written as follows ($\Lambda = 1,\dots,h_{2,1},h_{2,1}+1$ throughout[15]):

$$\begin{aligned}
K(t,\overline{t}) &= -ln\left[-i\left(t^0-\overline{t}^{\overline{0}}\right)\right] - ln\left[i\int_{CY_3}\Omega_3(x)\wedge\overline{\Omega}_3(\overline{x})\right] \\
&= -ln\left[\int_{CY_3}\left[t^0\Omega_3(x)\wedge\overline{\Omega}_3(\overline{x}) - \Omega_3(x)\wedge\overline{t}^{\overline{0}}\overline{\Omega}_3(\overline{x})\right]\right] \\
&= -ln\left[\left(t^0-\overline{t}^{\overline{0}}\right)\left(\overline{X}^\Lambda(\overline{x})F_\Lambda(x) - X^\Lambda(x)\overline{F}_\Lambda(\overline{x})\right)\right],
\end{aligned} \tag{201}$$

where Ω_3 is the holomorphic $(3,0)$-form defined on CY_3. Thus, one can write:

$$\begin{cases}
K(t,\overline{t}) = K_1\left(t^0,\overline{t}^{\overline{0}}\right) + K_3(x,\overline{x}); \\
K_1\left(t^0,\overline{t}^{\overline{0}}\right) \equiv -ln\left[-i\left(t^0-\overline{t}^{\overline{0}}\right)\right]; \\
K_3(x,\overline{x}) \equiv -ln\left[i\left(\overline{X}^\Lambda(\overline{x})F_\Lambda(x) - X^\Lambda(x)\overline{F}_\Lambda(\overline{x})\right)\right].
\end{cases} \tag{202}$$

The reality condition on K_1 and K_3 yields the conditions

$$Imt^0 > 0,\ Im\left(X^\Lambda(x)\overline{F}_\Lambda(\overline{x})\right) > 0. \tag{203}$$

The metric of the whole moduli space is given by ($a = 0,1,\dots,h_{2,1}$ throughout)

[15] Notice the different range of the symplectic (capital Greek) indices in the present treatment of Type IIB on CY_3 orientifold with O3/O7-planes with respect to the range $0,1,\dots,h_{2,1}$ of the previous treatment of Type IIB on CY_3. In general, the reference to the graviphoton degree of freedom "0" is lost, due to the orientifolding truncation of the low-energy, effective supergravity.

$$g_{a\bar{b}}(t,\bar{t}) = \overline{\partial}_{\bar{b}}\partial_a K(t,\bar{t}) = \overline{\partial}_{\bar{b}}\partial_a \left[K_1\left(t^0,\bar{t}^{\bar{0}}\right) + K_3(x,\bar{x})\right], \tag{204}$$

yielding

$$\begin{cases} g_{0\bar{0}} = -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2} = e^{2K_1\left(t^0,\bar{t}^{\bar{0}}\right)}; \\ g_{0\bar{i}} = 0 = g_{i\bar{0}}; \\ g_{i\bar{j}} = \overline{\partial}_{\bar{j}}\partial_i K_3(x,\bar{x}). \end{cases} \tag{205}$$

In our treatment, we will make extensive use of the local "flat" coordinates in M (denoted by capital indices $A = \underline{0}, \underline{1}, \ldots, \underline{h_{2,1}}$ throughout), defined as usual by $(g_{a\bar{b}}g^{a\bar{c}} = \delta_{\bar{b}}^{\bar{c}},\ g_{a\bar{b}}g^{c\bar{b}} = \delta_a^c)$

$$g_{a\bar{b}}(t,\bar{t}) \equiv e_a^A(t,\bar{t})\,\bar{e}_{\bar{b}}^{\bar{B}}(t,\bar{t})\,\delta_{A\bar{B}} \Leftrightarrow g^{a\bar{b}}(t,\bar{t}) \equiv e_A^a(t,\bar{t})\,\bar{e}_{\bar{B}}^{\bar{b}}(t,\bar{t})\,\delta^{A\bar{B}}, \tag{206}$$

where $e_a^A(t,\bar{t})$ is the local vielbein in M, and $e_A^a(t,\bar{t})$ is its inverse ($e_a^A e_A^b = \delta_a^b$, $e_a^A e_B^a = \delta_B^A$). Due to (205), the $h_{2,1}^2 + 2h_{2,1} + 1$ components of the vielbein $e_a^A = \left\{e_{0}^{\underline{0}}, e_{i}^{\underline{0}}, e_{0}^{I}, e_{i}^{I}\right\}$ ($I = \underline{1}, \ldots, \underline{h_{2,1}}$ throughout), defined by (206), satisfy the following set of Equations:

$$\begin{cases} \left|e_0^{\underline{0}}(t,\bar{t})\right|^2 + e_0^I(t,\bar{t})\,\bar{e}_{\bar{0}}^{\bar{J}}(t,\bar{t})\,\delta_{I\bar{J}} = -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2}; \\ e_0^{\underline{0}}(t,\bar{t})\,\bar{e}_{\bar{i}}^{\bar{\underline{0}}}(t,\bar{t}) + e_0^I(t,\bar{t})\,\bar{e}_{\bar{i}}^{\bar{J}}(t,\bar{t})\,\delta_{I\bar{J}} = 0; \\ e_i^{\underline{0}}(t,\bar{t})\,\bar{e}_{\bar{j}}^{\bar{\underline{0}}}(t,\bar{t}) + e_i^I(t,\bar{t})\,\bar{e}_{\bar{j}}^{\bar{J}}(t,\bar{t})\,\delta_{I\bar{J}} = \overline{\partial}_{\bar{j}}\partial_i K_3(x,\bar{x}), \end{cases} \tag{207}$$

admitting as a solution[16]:

$$\begin{cases} \left|e_0^{\underline{0}}(t,\bar{t})\right|^2 = -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2} \Leftarrow e_0^{\underline{0}}(t,\bar{t}) = \left(\bar{t}^{\bar{0}} - t^0\right)^{-1} = ie^{K_1\left(t^0,\bar{t}^{\bar{0}}\right)} = e_0^{\underline{0}}\left(t^0,\bar{t}^{\bar{0}}\right); \\ e_0^I(t,\bar{t}) = 0,\ \forall I = \underline{1}, \ldots, \underline{h_{2,1}}; \\ e_i^{\underline{0}}(t,\bar{t}) = 0,\ \forall i = 1, \ldots, h_{2,1}; \\ e_i^I(t,\bar{t})\,\bar{e}_{\bar{j}}^{\bar{J}}(t,\bar{t})\,\delta_{I\bar{J}} = \overline{\partial}_{\bar{j}}\partial_i K_3(x,\bar{x}). \end{cases} \tag{208}$$

By inverting (206) one gets

$$\delta_{A\bar{B}} = e_A^a(t,\bar{t})\,\bar{e}_{\bar{B}}^{\bar{b}}(t,\bar{t})\,g_{a\bar{b}}(t,\bar{t}) \Leftrightarrow \delta^{A\bar{B}} = e_a^A(t,\bar{t})\,\bar{e}_{\bar{b}}^{\bar{B}}(t,\bar{t})\,g^{a\bar{b}}(t,\bar{t}), \tag{209}$$

[16] Notice that the solutions given by (208) and (211) are clearly not unique. Indeed, for a given metric, one can always transform the vielbein and its inverse by a Lorentz transformation, which however will not affect the metric itself.

which by (205) implies that the $h_{2,1}^2 + 2h_{2,1} + 1$ components of the inverse vielbein $e_A^a = \left\{ e_{\underline{0}}^0, e_{\underline{0}}^i, e_I^0, e_I^i \right\}$, defined by (206), satisfy the set following set of Equations:

$$\begin{cases} -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2} \left| e_{\underline{0}}^0(t,\bar{t}) \right|^2 + e_{\underline{0}}^i(t,\bar{t})\, \overline{e}_{\underline{0}}^{\bar{j}}(t,\bar{t})\, \overline{\partial}_{\bar{j}} \partial_i K_3(x,\bar{x}) = 1; \\ -\left(\bar{t}^{\bar{0}} - t^0\right)^{-1} \overline{e}_{\bar{I}}^{\bar{0}}(t,\bar{t}) + e_{\underline{0}}^i(t,\bar{t})\, \overline{e}_{\bar{I}}^{\bar{j}}(t,\bar{t})\, \overline{\partial}_{\bar{j}} \partial_i K_3(x,\bar{x}) = 0; \\ -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2} e_I^0(t,\bar{t})\, \overline{e}_{\bar{J}}^{\bar{0}}(t,\bar{t}) + e_I^i(t,\bar{t})\, \overline{e}_{\bar{J}}^{\bar{j}}(t,\bar{t})\, \overline{\partial}_{\bar{j}} \partial_i K_3(x,\bar{x}) = \delta_{I\bar{J}}, \end{cases} \tag{210}$$

admitting as a solution:

$$\begin{cases} \left| e_{\underline{0}}^0(t,\bar{t}) \right|^2 = -\left(\bar{t}^{\bar{0}} - t^0\right)^2 = \left| e_{0}^{\underline{0}}(t,\bar{t}) \right|^{-2}; \\ \Uparrow \\ e_{\underline{0}}^0(t,\bar{t}) = \left(\bar{t}^{\bar{0}} - t^0\right) = -ie^{-K_1\left(t^0,\bar{t}^{\bar{0}}\right)} = \left[e_0^{\underline{0}}\left(t^0,\bar{t}^{\bar{0}}\right) \right]^{-1} = e_{\underline{0}}^0\left(t^0,\bar{t}^{\bar{0}}\right); \\ e_{\underline{0}}^i(t,\bar{t}) = 0,\ \forall i = 1,\ldots,h_{2,1}; \\ e_I^0(t,\bar{t}) = 0,\ \forall I = \underline{1},\ldots,\underline{h_{2,1}}; \\ e_I^i(t,\bar{t})\, \overline{e}_{\bar{J}}^{\bar{j}}(t,\bar{t})\, \overline{\partial}_{\bar{j}} \partial_i K_3(x,\bar{x}) = \delta_{I\bar{J}}, \end{cases} \tag{211}$$

implying, by (206), that the components of the inverse metric tensor of M read as follows:

$$\begin{cases} g^{0\bar{0}} = -\left(\bar{t}^{\bar{0}} - t^0\right)^2 = e^{-2K_1\left(t^0,\bar{t}^{\bar{0}}\right)} = \left(g_{0\bar{0}}\right)^{-1}; \\ g^{0\bar{i}} = 0 = g^{i\bar{0}}; \\ g^{i\bar{j}} : g^{i\bar{j}} \overline{\partial}_{\bar{j}} \partial_k K_3(x,\bar{x}) = \delta_k^i,\ g^{i\bar{j}} \overline{\partial}_{\bar{k}} \partial_i K_3(x,\bar{x}) = \delta_{\bar{k}}^{\bar{j}}. \end{cases} \tag{212}$$

Moreover, it should be noticed that actually $e_i^I = e_i^I(x,\bar{x})$ and $e_I^i = e_I^i(x,\bar{x})$, as obtained by differentiating with respect to the axion-dilaton t^0 the fourth Equation of the systems of solutions (208) and (211), respectively:

$$\begin{gathered} \left\{ \left[\partial_0 e_i^I(t,\bar{t}) \right] \overline{e}_{\bar{j}}^{\bar{J}}(t,\bar{t}) + e_i^I(t,\bar{t})\, \partial_0 \overline{e}_{\bar{j}}^{\bar{J}}(t,\bar{t}) \right\} \delta_{I\bar{J}} = 0; \\ \Updownarrow \\ \begin{cases} \partial_0 e_i^I(t,\bar{t}) = 0, \\ \partial_0 \overline{e}_{\bar{i}}^{\bar{I}}(t,\bar{t}) = 0 \Leftrightarrow \overline{\partial}_{\bar{0}} e_i^I(t,\bar{t}) = 0; \end{cases} \\ \Updownarrow \\ e_i^I = e_i^I(x,\bar{x}); \end{gathered} \tag{213}$$

$$\left\{ \left[\partial_0 e_I^i(t,\bar{t})\right] \overline{e}_{\overline{J}}^{\overline{j}}(t,\bar{t}) + e_I^i(t,\bar{t})\,\partial_0 \overline{e}_{\overline{J}}^{\overline{j}}(t,\bar{t}) \right\} \overline{\partial}_{\overline{j}}\partial_i K_3(x,\bar{x}) = 0;$$

$$\Updownarrow$$

$$\begin{cases} \partial_0 e_I^i(t,\bar{t}) = 0, \\ \\ \partial_0 \overline{e}_{\overline{I}}^{\overline{i}}(t,\bar{t}) \Leftrightarrow \overline{\partial}_{\overline{0}} e_I^i(t,\bar{t}) = 0; \end{cases}$$

$$\Updownarrow$$

$$e_I^i = e_I^i(x,\bar{x}).$$

In the following treatment, we will use the solutions (208) and (211) of the systems of (207) and (210), respectively, i.e. we will assume that a system of local "flat" coordinates in M defined by (206) and (209) always exists such that the corresponding vielbein and its inverse are given by (208) and (211) (implemented by (4.1.1)), in turn consistent with the covariant and contravariant metric tensor of M given by (205) and (212), respectively.

4.1.2 1-, 3- and 4-Forms on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$

Next, we introduce the Ramond-Ramond (RR) and Neveu-Schwarz-Neveu-Schwarz (NSNS) flux 3-forms of Type IIB on CY_3 orientifold (with O3/O7-planes) as follows:

$$\begin{aligned} &RR : \mathfrak{F}_3 \equiv p_f^\Lambda \alpha_\Lambda - q_{f|\Lambda}\beta^\Lambda \in H^3(CY_3,\mathbb{R}); \\ &NSNS : \mathfrak{H}_3 \equiv p_h^\Lambda \alpha_\Lambda - q_{h|\Lambda}\beta^\Lambda \in H^3(CY_3,\mathbb{R}), \end{aligned} \tag{214}$$

where we introduced the $1\times(2h_{2,1}+2)$ symplectic vector of RR and NSNS fluxes (charges), respectively:

$$\begin{aligned} &Q_{RR} \equiv \left(p_f^\Lambda, q_{f|\Lambda}\right); \\ &Q_{NSNS} \equiv \left(p_h^\Lambda, q_{h|\Lambda}\right), \end{aligned} \tag{215}$$

and $\{\alpha_\Lambda,\beta^\Lambda\}$ is the b_3-dim. real (manifestly symplectic-covariant) basis of the third real cohomology $H^3(CY_3,\mathbb{R})$, satisfying (159). In the considered framework, the flux 3-forms defined by (214) can be unified in the t^0-dependent, complex flux 3-form

$$\mathfrak{G}_3\left(t^0\right) \equiv \mathfrak{F}_3 - t^0\mathfrak{H}_3 = \left(p_f^\Lambda - t^0 p_h^\Lambda\right)\alpha_\Lambda - \left(q_{f|\Lambda} - t^0 q_{h|\Lambda}\right)\beta^\Lambda \in H^3\left(CY_3,\mathbb{C};t^0\right), \tag{216}$$

thus determining the GVW $\mathcal{N}=1,\ d=4$ holomorphic superpotential as follows:

$$\begin{aligned} W(t) &\equiv \int_{CY_3}\mathfrak{G}_3\left(t^0\right)\wedge\Omega_3(x) = \int_{CY_3}\mathfrak{F}_3\wedge\Omega_3(x) - t^0\int_{CY_3}\mathfrak{H}_3\wedge\Omega_3(x) \\ &= q_{f|\Lambda}X^\Lambda(x) - p_f^\Lambda F_\Lambda(x) + q_{h|\Lambda}\left(-t^0X^\Lambda(x)\right) - p_h^\Lambda\left(-t^0F_\Lambda(x)\right). \end{aligned} \tag{217}$$

The second line of (217) suggests to redefine the holomorphic $(3,0)$-form in the *"NSNS sector"* as follows:

$$\Omega_{3,NS}(t) \equiv -t^0\Omega_{3,RR}(x) = -t^0\Omega_3(x). \tag{218}$$

Since in Type IIB on the considered CY_3 orientifold the flux 3-forms $\mathfrak{F}_3$ and $\mathfrak{H}_3$ form the $SL\left(2,H^3\left(CY_3,\mathbb{R}\right)\right)$-doublet

$$\widehat{F} \equiv \begin{pmatrix} \mathfrak{F}_3 \\ \\ \mathfrak{H}_3 \end{pmatrix} \in SL\left(2,H^3\left(CY_3,\mathbb{R}\right)\right), \tag{219}$$

correspondingly, one can introduce the $SL\left(2,H^{3,0}\left(CY_3;t\right)\right)$-doublet

$$\Xi(t) \equiv \begin{pmatrix} \Xi_1(x) \equiv \Omega_3(x) \\ \\ \Xi_2(t) \equiv -t^0\Omega_3(x) \end{pmatrix} \in SL\left(2,H^{3,0}\left(CY_3;t\right)\right). \tag{220}$$

By exploiting such a manifest $SL(2)$-covariance, (201) and (217) can be rewritten as follows:

$$K(t,\bar{t}) = -ln\left[\int_{CY_3}\left[\Xi_1(x)\wedge\overline{\Xi}_2(\bar{t}) - \Xi_2(t)\wedge\overline{\Xi}_1(\bar{x})\right]\right]; \tag{221}$$

$$W(t) = \int_{CY_3}\left[\mathfrak{F}_3\wedge\Xi_1(x) + \mathfrak{H}_3\wedge\Xi_2(t)\right] = \int_{CY_3}\widehat{F}^T\wedge\Xi(t). \tag{222}$$

Thus, the $\mathscr{N}=1,\ d=4$ covariantly holomorphic central charge function of Type IIB on CY_3 orientifold with O3/O7-planes can be introduced:

$$Z(t,\bar{t}) \equiv e^{\frac{1}{2}K(t,\bar{t})}W(t) = e^{\frac{1}{2}K(t,\bar{t})}\int_{CY_3}\mathfrak{G}_3\left(t^0\right)\wedge\Omega_3(x) \tag{223}$$

$$= \frac{\int_{CY_3}\widehat{F}^T\wedge\Xi(t)}{\sqrt{\int_{CY_3}\left[\Xi_1(x)\wedge\overline{\Xi}_2(\bar{t}) - \Xi_2(t)\wedge\overline{\Xi}_1(\bar{x})\right]}} \tag{224}$$

$$= \frac{\left(q_{f|\Lambda} - t^0 q_{h|\Lambda}\right)X^\Lambda(x) - \left(p_f^\Lambda - t^0 p_h^\Lambda\right)F_\Lambda(x)}{\sqrt{\left(t^0-\overline{t}^0\right)\left(\overline{X}^\Lambda(\bar{x})F_\Lambda(x) - X^\Lambda(x)\overline{F}_\Lambda(\bar{x})\right)}}, \tag{225}$$

with Kähler weights $(1,-1)$ with respect to $K(t,\bar{t})$:

$$\begin{aligned} D_a Z(t,\bar{t}) &= \partial_a Z(t,\bar{t}) + \frac{1}{2}\left(\partial_a K(t,\bar{t})\right)Z(t,\bar{t}); \\ \overline{D}_{\bar{a}} Z(t,\bar{t}) &= \overline{\partial}_{\bar{a}} Z(t,\bar{t}) - \frac{1}{2}\left(\overline{\partial}_{\bar{a}} K(t,\bar{t})\right)Z(t,\bar{t}) = 0. \end{aligned} \tag{226}$$

Now, we can perform an unifying simplification of notation by using the language of 4-forms on Calabi-Yau 4-folds (CY_4); in such a framework, Type IIB on CY_3 orientifold with O3/O7-planes can be described by 4-forms defined on $CY_4 = \frac{CY_3 \times T^2}{\mathbb{Z}_2}$, where T^2 denotes the "auxiliary" 2-torus, whose complex modulus is the universal-axion dilaton $\tau \equiv t^0$. Thus, beside the b_3-dim. real (manifestly symplectic-covariant) basis $\left\{\alpha_\Lambda,\ \beta^\Lambda\right\}$ of $H^3\left(CY_3, \mathbb{R}\right)$ (satisfying (159)), one can introduce the 2-dim. basis $\{\alpha,\ \beta\}$ of $H^1\left(T^2, \mathbb{R}\right)$, satisfying

$$\int_{T^2} \alpha \wedge \alpha = 0 = \int_{T^2} \beta \wedge \beta,\ \int_{T^2} \alpha \wedge \beta = 1, \tag{227}$$

and the holomorphic $(1,0)$-form $\Omega_1\left(t^0\right)$ on T^2:

$$\Omega_1\left(t^0\right) \equiv -t^0 \alpha + \beta \in H^{1,0}\left(T^2\right). \tag{228}$$

By recalling (161), it is thus possible to define an holomorphic $(4,0)$-form on $CY_4 (= \frac{CY_3 \times T^2}{\mathbb{Z}_2}$, as always understood in treatment below) as follows:

$$\begin{aligned} \Omega_4(t) \equiv \Omega_1\left(t^0\right) \wedge \Omega_3(x) &= X^\Lambda(x)\, \beta \wedge \alpha_\Lambda - t^0 X^\Lambda(x)\, \alpha \wedge \alpha_\Lambda \\ &\quad - F_\Lambda(x)\, \beta \wedge \beta^\Lambda + t^0 F_\Lambda(x)\, \alpha \wedge \beta^\Lambda. \end{aligned} \tag{229}$$

Instead of using the complex, t^0-dependent flux 3-form $\mathfrak{G}_3\left(t^0\right) \in H^3\left(CY_3, \mathbb{C}; t^0\right)$ defined by (4.1.2.3), the RR and NSNS flux 3-forms can be unified elegantly by introducing the real flux 4-form

$$\begin{aligned} \mathfrak{F}_4 \equiv -\alpha \wedge \mathfrak{F}_3 + \beta \wedge \mathfrak{H}_3 &= \left(p_h^\Lambda \beta - p_f^\Lambda \alpha\right) \wedge \alpha_\Lambda \\ &\quad - \left(q_{h|\Lambda} \beta - q_{f|\Lambda} \alpha\right) \wedge \beta^\Lambda \in H^4\left(CY_4, \mathbb{R}\right). \end{aligned} \tag{230}$$

By using (159), (161), (227), (228), (229) and (230), (221), (222), (223), (224) can be elegantly rewritten as follows:

$$K(t, \bar{t}) = -ln\left(\int_{CY_4} \Omega_4(t) \wedge \overline{\Omega}_4(\bar{t})\right); \tag{231}$$

$$W(t) = \int_{CY_4} \mathfrak{F}_4 \wedge \Omega_4(t); \tag{232}$$

$$Z(t, \bar{t}) = e^{\frac{1}{2}K(t, \bar{t})} \int_{CY_4} \mathfrak{F}_4 \wedge \Omega_4(t) = \frac{\int_{CY_4} \mathfrak{F}_4 \wedge \Omega_4(t)}{\sqrt{\int_{CY_4} \Omega_4(t) \wedge \overline{\Omega}_4(\bar{t})}} = \int_{CY_4} \mathfrak{F}_4 \wedge \hat{\Omega}_4(t, \bar{t}), \tag{233}$$

where in (233) we defined the covariantly holomorphic 4-form on CY_4:

$$\begin{aligned}\hat{\Omega}_4(t,\bar{t}) &\equiv e^{\frac{1}{2}K(t,\bar{t})}\Omega_4(t) = e^{\frac{1}{2}K_1\left(t^0,\bar{t}^{\bar{0}}\right)}e^{\frac{1}{2}K_3(x,\bar{x})}\Omega_1\left(t^0\right)\wedge\Omega_3(x)\\ &= \hat{\Omega}_1\left(t^0,\bar{t}^{\bar{0}}\right)\wedge\hat{\Omega}_3(x,\bar{x});\ \hat{\Omega}_1\left(t^0,\bar{t}^{\bar{0}}\right)\equiv e^{\frac{1}{2}K_1\left(t^0,\bar{t}^{\bar{0}}\right)}\Omega_1\left(t^0\right);\end{aligned} \tag{234}$$

$\hat{\Omega}_3(x,\bar{x})$ is the covariantly holomorphic 3-form on CY_3, defined by (190); it has Kähler weights $(1,-1)$ with respect to the Kähler potential $K_3(x,\bar{x})$ of the SK CS moduli space $\mathscr{M}_{CS}$ of CY_3:

$$\begin{aligned}
&D_i\hat{\Omega}_3 = \begin{cases}\partial_i\hat{\Omega}_3+\frac{1}{2}(\partial_iK_3)\hat{\Omega}_3 = e^{\frac{1}{2}K_3}D_i\Omega_3\\[2mm] = \dfrac{1}{\sqrt{i\left(\overline{X}^{\Delta}F_\Delta - X^\Delta\overline{F}_\Delta\right)}}\left\{\begin{array}{l}\left[\partial_iX^\Lambda-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_\Sigma-\left(\partial_iX^\Sigma\right)\overline{F}_\Sigma\right)}{\overline{X}^{\Xi}F_\Xi-X^\Xi\overline{F}_\Xi}X^\Lambda\right]\alpha_\Lambda+\\[4mm] -\left[\partial_iF_\Lambda-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_\Sigma-\left(\partial_iX^\Sigma\right)\overline{F}_\Sigma\right)}{\overline{X}^{\Xi}F_\Xi-X^\Xi\overline{F}_\Xi}F_\Lambda\right]\beta^\Lambda\end{array}\right\};\end{cases}\\
&\overline{D}_{\bar{i}}\hat{\Omega}_3 = \overline{\partial}_{\bar{i}}\hat{\Omega}_3-\frac{1}{2}\left(\overline{\partial}_{\bar{i}}K_3\right)\hat{\Omega}_3 = 0;\\
&D_iD_j\hat{\Omega}_3 = iC_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_3;\ \overline{D}_{\bar{i}}D_j\hat{\Omega}_3 = g_{j\bar{i}}\hat{\Omega}_3;\\
&D_0\hat{\Omega}_3 = 0;\ \overline{D}_{\bar{0}}\hat{\Omega}_3 = 0.
\end{aligned} \tag{235}$$

On the other hand, $\hat{\Omega}_1\left(t^0,\bar{t}^{\bar{0}}\right)$ is the covariantly holomorphic 1-form on T^2, defined by the second line of (234); it has Kähler weights $(1,-1)$ with respect to the Kähler potential $K_1\left(t^0,\bar{t}^{\bar{0}}\right)$ of the Kähler 1-dim. moduli space $\mathscr{M}_{t^0}$ of T^2, spanned by the universal axion-dilaton $\tau\equiv t^0$:

$$\begin{aligned}
&D_0\hat{\Omega}_1 = \partial_0\hat{\Omega}_1+\frac{1}{2}(\partial_0K_1)\hat{\Omega}_1 = e^{\frac{1}{2}K_1}D_0\Omega_1 = ie^{K_1}\overline{\hat{\Omega}}_1 = \left(\bar{t}^{\bar{0}}-t^0\right)^{-1}\overline{\hat{\Omega}}_1;\\
&\qquad\Updownarrow\\
&\overline{\hat{\Omega}}_1 = -ie^{-K_1}D_0\hat{\Omega}_1 \Leftrightarrow \hat{\Omega}_1 = ie^{-K_1}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_1 \Leftrightarrow \overline{D}_{\bar{0}}\overline{\hat{\Omega}}_1 = -ie^{K_1}\hat{\Omega}_1;\\
&\overline{D}_{\bar{0}}\hat{\Omega}_1 = \overline{\partial}_{\bar{0}}\hat{\Omega}_1-\frac{1}{2}\left(\overline{\partial}_{\bar{0}}K_1\right)\hat{\Omega}_1 = 0;\\
&D_0D_0\hat{\Omega}_1 = 0;\ \overline{D}_{\bar{0}}D_0\hat{\Omega}_1 = g_{0\bar{0}}\hat{\Omega}_1 = e^{2K_1}\hat{\Omega}_1 = -\left(\bar{t}^{\bar{0}}-t^0\right)^{-2}\hat{\Omega}_1;\\
&D_i\hat{\Omega}_1 = 0;\ \overline{D}_{\bar{i}}\hat{\Omega}_1 = 0.
\end{aligned} \tag{236}$$

Resultingly, the covariantly holomorphic 4-form $\hat{\Omega}_4(t,\bar{t})$ on $CY_4 = \frac{CY_3\times T^2}{\mathbb{Z}_2}$, defined by the first line of (234), has Kähler weights $(1,-1)$ with respect to the whole

Kähler potential $K(t,\bar{t}) = K_1\left(t^0,\overline{t^0}\right) + K_3(x,\bar{x})$ of the $(h_{2,1}+1)$-dim. moduli space $M = \mathscr{M}_{t^0} \otimes \mathscr{M}_{CS}$. of CY_4 (recall (200)):

$$\begin{aligned} D_a\hat{\Omega}_4(t,\bar{t}) &= \partial_a\hat{\Omega}_4(t,\bar{t}) + \frac{1}{2}\left(\partial_a K(t,\bar{t})\right)\hat{\Omega}_4(t,\bar{t}); \\ \overline{D}_{\bar{a}}\hat{\Omega}_4(t,\bar{t}) &= \overline{\partial}_{\bar{a}}\hat{\Omega}_4(t,\bar{t}) - \frac{1}{2}\left(\overline{\partial}_{\bar{a}} K(t,\bar{t})\right)\hat{\Omega}_4(t,\bar{t}) = 0, \end{aligned} \tag{237}$$

implying that

$$D_b\overline{D}_{\bar{a}}\hat{\Omega}_4(t,\bar{t}) = g_{b\bar{a}}\hat{\Omega}_4(t,\bar{t}). \tag{238}$$

4.1.3 Hodge Decomposition of $\mathfrak{F}_4$

Now, in order to derive the Hodge-decomposition[17] of the real flux 4-form $\mathfrak{F}_4$, we have to determine all the possible independent 4-forms on $CY_4(=\frac{CY_3\times T^2}{\mathbb{Z}_2}$, as always understood throughout). Due to (237) and (238), it is easy to realize that, up to the third order of covariant differentiation included, the possible independent 4-forms ($(1,-1)$-Kähler weighted with respect to K) on CY_4 are $\hat{\Omega}_4$, $D_a\hat{\Omega}_4$, $D_aD_b\hat{\Omega}_4$, $D_aD_bD_c\hat{\Omega}_4$ and $\overline{D}_{\bar{a}}D_bD_c\hat{\Omega}_4$.

As can be realized by considering (269), (270), (271), (272) of Appendix I, $D_aD_b\hat{\Omega}_4$ cannot be expressed in terms of $D_a\hat{\Omega}_4$ (as instead it happens in the extremal BH case, see (177)), and all the independent, $(1,-1)$-Kähler-weighted 4-forms on the considered CY_4 are given by the $2h_{2,1}+2$ forms

$$\hat{\Omega}_4,\ D_0\hat{\Omega}_4,\ D_i\hat{\Omega}_4,\ D_0D_i\hat{\Omega}_4. \tag{239}$$

The third covariant derivatives of $\hat{\Omega}_4$ do not add any other independent 4-form, and so also all the other higher order covariant derivatives of $\hat{\Omega}_4$. Thus, the possible candidates along which one might decompose the real flux 4-form $\mathfrak{F}_4$ are the 4-forms given by (239) and their complex conjugated $\overline{\hat{\Omega}}_4$, $\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4$, $\overline{D}_{\bar{i}}\overline{\hat{\Omega}}_4$, $\overline{D}_{\bar{0}}\overline{D}_{\bar{i}}\overline{\hat{\Omega}}_4$.

The *"intersections"* among the elements of the set of 4-forms $\hat{\Omega}_4$, $D_0\hat{\Omega}_4$, $D_i\hat{\Omega}_4$, $D_0D_i\hat{\Omega}_4$, $\overline{\hat{\Omega}}_4$, $\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4$, $\overline{D}_{\bar{i}}\overline{\hat{\Omega}}_4$ and $\overline{D}_{\bar{0}}\overline{D}_{\bar{i}}\overline{\hat{\Omega}}_4$ in generic local "curved" and in local "flat" coordinates of M are given in Appendix II. By using such results, the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ can be thus Hodge-decomposed as follows $(\eta_1,\ldots,\eta_6 \in \mathbb{C})$

$$\mathfrak{F}_4 = \left[\begin{array}{l} \eta_1\left(\int_{CY_4}\mathfrak{F}_4\wedge\hat{\Omega}_4\right)\overline{\hat{\Omega}}_4 + \eta_2\delta^{A\overline{B}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\left(D_A\hat{\Omega}_4\right)\right)\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4 + \\ +\eta_3\delta^{A\overline{B}}\left(\int_{CY_4}\mathfrak{F}_4\wedge D_{\underline{0}}D_A\hat{\Omega}_4\right)\overline{D}_{\underline{\bar{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4 + \eta_4\delta^{B\overline{A}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\overline{D}_{\underline{\bar{0}}}\overline{D}_{\overline{A}}\overline{\hat{\Omega}}_4\right)D_{\underline{0}}D_B\hat{\Omega}_4 + \\ +\eta_5\delta^{B\overline{A}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\left(\overline{D}_{\overline{A}}\overline{\hat{\Omega}}_4\right)\right)D_B\hat{\Omega}_4 + \eta_6\left(\int_{CY_4}\mathfrak{F}_4\wedge\overline{\hat{\Omega}}_4\right)\hat{\Omega}_4 \end{array}\right] \tag{240}$$

[17] For an elegant and detailed derivation of the Hodge-decomposition of $\mathfrak{F}_4$ using methods of algebraic geometry, see e.g. Sect. 2 of [93].

$$= \begin{bmatrix} \eta_1 Z\overline{\hat{\Omega}}_4 + \eta_2 \delta^{A\overline{B}} (D_A Z) \overline{D}_{\overline{B}} \overline{\hat{\Omega}}_4 + \eta_3 \delta^{A\overline{B}} (D_{\underline{0}} D_A Z) \overline{D}_{\overline{\underline{0}}} \overline{D}_{\overline{B}} \overline{\hat{\Omega}}_4 + \\ \\ +\eta_4 \delta^{B\overline{A}} \left(\overline{D}_{\overline{\underline{0}}} \overline{D}_{\overline{A}} \overline{Z}\right) D_{\underline{0}} D_B \hat{\Omega}_4 + \eta_5 \delta^{B\overline{A}} \left(\overline{D}_{\overline{A}} \overline{Z}\right) D_B \hat{\Omega}_4 + \eta_6 \overline{Z} \hat{\Omega}_4 \end{bmatrix}, \tag{241}$$

where (233) was used, also implying:

$$\begin{aligned} &\int_{CY_4} \mathfrak{F}_4 \wedge D_a \hat{\Omega}_4 = D_a Z, \quad \int_{CY_4} \mathfrak{F}_4 \wedge D_a D_b \hat{\Omega}_4 = D_a D_b Z; \\ &\int_{CY_4} \mathfrak{F}_4 \wedge D_A \hat{\Omega}_4 = D_A Z, \quad \int_{CY_4} \mathfrak{F}_4 \wedge D_A D_B \hat{\Omega}_4 = D_A D_B Z. \end{aligned} \tag{242}$$

The r.h.s. of the Hodge-decomposition (241) is the most general Kähler gauge-invariant combination of *all* the possible $((1,-1)$ and $(-1,1))$-Kähler-weighted independent 4-forms for Type IIB on $CY_4 = \frac{CY_3 \times T^2}{\mathbb{Z}_2}$. The reality condition $\overline{\mathfrak{F}}_4 = \mathfrak{F}_4$ implies $\eta_4 = \overline{\eta_3}$, $\eta_5 = \overline{\eta_2}$ and $\eta_6 = \overline{\eta_1}$. The (a priori) complex coefficients η_1, η_2 and η_3 can be determined by computing $\int_{CY_4} \mathfrak{F}_4 \wedge \hat{\Omega}_4$, $\int_{CY_4} \mathfrak{F}_4 \wedge D_A \hat{\Omega}_4$ and $\int_{CY_4} \mathfrak{F}_4 \wedge D_{\underline{0}} D_A \hat{\Omega}_4$, and using the identity (241) and recalling (233), (242) and the *"intersections"* in local "flat" coordinates (281), (282), (283), (284). By doing so, one obtains:

$$\begin{aligned} Z &= \int_{CY_4} \mathfrak{F}_4 \wedge \hat{\Omega}_4 = \eta_1 Z \int_{CY_4} \overline{\hat{\Omega}}_4 \wedge \hat{\Omega}_4 = \eta_1 Z \Leftrightarrow \eta_1 = 1; \\ D_C Z &= \int_{CY_4} \mathfrak{F}_4 \wedge D_C \hat{\Omega}_4 = \eta_2 \delta^{A\overline{B}} (D_A Z) \int_{CY_4} \left(\overline{D}_{\overline{B}} \overline{\hat{\Omega}}_4\right) \wedge D_C \hat{\Omega}_4 \\ &= -\eta_2 \delta^{A\overline{B}} D_A Z \delta_{C\overline{B}} = -\eta_2 D_C Z \Leftrightarrow \eta_2 = -1; \\ D_{\underline{0}} D_C Z &= \int_{CY_4} \mathfrak{F}_4 \wedge D_{\underline{0}} D_C \hat{\Omega}_4 = \eta_3 \delta^{A\overline{B}} \left(D_{\underline{0}} D_A Z\right) \int_{CY_4} \left(\overline{D}_{\overline{\underline{0}}} \overline{D}_{\overline{B}} \overline{\hat{\Omega}}_4\right) \wedge D_{\underline{0}} D_C \hat{\Omega}_4 \\ &= \eta_3 \delta^{A\overline{B}} \delta_{C\overline{B}} D_{\underline{0}} D_A Z = \eta_3 D_{\underline{0}} D_C Z \Leftrightarrow \eta_3 = 1. \end{aligned} \tag{243}$$

Thus, the complete Hodge-decomposition of the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ of Type IIB on $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$ in generic local "flat" coordinates[18] in M reads

$$\mathfrak{F}_4 = 2Re\left[\overline{Z}\hat{\Omega}_4 - \left(\overline{D}^A \overline{Z}\right) D_A \hat{\Omega}_4 + \left(\overline{D}^{\underline{0}} \overline{D}^A \overline{Z}\right) D_{\underline{0}} D_A \hat{\Omega}_4\right] \tag{244}$$

$$= 2Re \begin{bmatrix} \overline{Z}\hat{\Omega}_1 \wedge \hat{\Omega}_3 - \left(\overline{D}_{\overline{\underline{0}}} \overline{Z}\right) \overline{\hat{\Omega}}_1 \wedge \hat{\Omega}_3 - \left(\overline{D}^I \overline{Z}\right) \hat{\Omega}_1 \wedge D_I \hat{\Omega}_3 + \\ \\ + \left(\overline{D}^{\underline{0}} \overline{D}^I \overline{Z}\right) \overline{\hat{\Omega}}_1 \wedge D_I \hat{\Omega}_3 \end{bmatrix}. \tag{245}$$

[18] For the analogous expression in generic local "curved" coordinates in M, see (285), (286), (287), (288).

4.2 $\mathcal{N}$ = 1, d = 4 *Effective Potential and* "Criticality Conditions" Approach

The potential of $\mathcal{N}=1,\ d=4$ supergravity (from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$), which acts as effective potential for the FV attractors, is given by (17), which we repeat here [88, 89]:

$$\begin{aligned} V_{\mathcal{N}=1} &= e^K\left[-3\left|W\right|^2+g^{a\overline{b}}D_aW\overline{D}_{\overline{b}}\overline{W}\right]=-3\left|Z\right|^2+g^{a\overline{b}}D_aZ\overline{D}_{\overline{b}}\overline{Z} \\ &= -3\left|Z\right|^2-\left(\overline{t}^{\overline{0}}-t^0\right)^2\left|D_0Z\right|^2+g^{i\overline{j}}\left(t^k,\overline{t}^{\overline{k}}\right)D_iZ\overline{D}_{\overline{j}}\overline{Z}\gtreqless 0. \end{aligned} \tag{246}$$

At a glance, the first difference between the "BH effective potential" V_{BH} given by (48) and the "FV effective potential" $V_{\mathcal{N}=1}$ given by (246) concerns their sign. Indeed, V_{BH} is positive-definite and it can be recognized as the first, quadratic invariant of SK geometry; through the Bekenstein-Hawking entropy-area formula, it is related to the classical entropy and to the area of the event horizon of the considered extremal (static, spherically symmetric, asymptotically flat) BH. On the other hand, $V_{\mathcal{N}=1}$ does not have a definite sign, and critical points of $V_{\mathcal{N}=1}$ can exist with $V_{\mathcal{N}=1}\gtreqless 0$:

(1) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0}>0$ corresponds to De Sitter (dS) vacua;
(2) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0}=0$ determines Minkowski vacua;
(3) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0}<0$ corresponds to anti De Sitter (AdS) vacua.

By differentiating (246) with respect to the moduli and recalling (232) and (238), one obtains the general criticality conditions of $V_{\mathcal{N}=1}$ ($\forall a=0,1,\ldots,h_{2,1}$):

$$\begin{gathered} D_aV_{\mathcal{N}=1}=\partial_aV_{\mathcal{N}=1}=0; \\ \Updownarrow \\ e^K\left[-3\overline{W}D_aW+g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W}+g^{b\overline{c}}\left(D_bW\right)D_a\overline{D}_{\overline{c}}\overline{W}\right] \\ =e^K\left[-2\overline{W}D_aW+g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W}\right]=0; \\ \Updownarrow \\ -2\overline{W}D_aW+g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W}=0, \end{gathered} \tag{247}$$

where, as in the case of extremal BH attractors in $\mathcal{N}=2,\ d=4$ supergravity, we assumed the Kähler potential to be regular, i.e. that $|K|<\infty$ globally in M (or *at least* at the critical points of $V_{\mathcal{N}=1}$).

Equation (247) are the what one should rigorously refer to as the $\mathcal{N}=1,\ d=4$ FV AEs (in Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$). By recalling (212), they can finally be rewritten as

$$\begin{aligned} D_aV_{\mathcal{N}=1} = \partial_aV_{\mathcal{N}=1}=0 \Leftrightarrow -2\overline{W}D_aW+g^{0\overline{0}}\left(D_aD_0W\right)\overline{D}_{\overline{0}}\overline{W} \\ +g^{i\overline{j}}\left(D_aD_iW\right)\overline{D}_{\overline{j}}\overline{W}=0,\forall a=0,1,\ldots,h_{2,1}. \end{aligned} \tag{248}$$

Let us specify such FV AEs for the two classes of (local "curved") indices; with some elaborations, one obtains:

$$a = 0 \text{ (axion-dilaton direction in } M) : \\ D_0 V_{\mathcal{N}=1} = \partial_0 V_{\mathcal{N}=1} = 0 \Leftrightarrow -2\overline{W} D_0 W + g^{j\bar{k}} (D_0 D_j W) \overline{D}_{\bar{k}} \overline{W} = 0; \tag{249}$$

$a = i \in \{1, \ldots, h_{2,1}\}$ (CS directions in M) :

$$\begin{gathered} D_i V_{\mathcal{N}=1} = \partial_i V_{\mathcal{N}=1} = 0 \\ \Updownarrow \\ -2\overline{W} D_i W + g^{0\bar{0}} (D_i D_0 W) \overline{D}_{\bar{0}} \overline{W} + g^{j\bar{k}} (D_i D_j W) \overline{D}_{\bar{k}} \overline{W} = 0; \\ \Updownarrow \\ -2\overline{W} D_i W + e^{-2K_1} (D_0 D_i W) \overline{D}_{\bar{0}} \overline{W} - e^{-K_1} g^{j\bar{k}} C_{ijl} g^{l\bar{m}} \left(\overline{D}_{\bar{0}} \overline{D}_{\bar{m}} \overline{W} \right) \overline{D}_{\bar{k}} \overline{W} = 0. \end{gathered} \tag{250}$$

Thus, despite the presence of the universal axion-dilaton direction in the $(h_{2,1}+1)$-dim. Kähler moduli space M, (250) yields that the tensor C_{ijk}, defined in the $h_{2,1}$-dim. SK CS moduli space $\mathcal{M}_{CS} \subsetneq M$, still plays a key role. The FV AEs (249) and (250) of $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$ relate, at the critical points of the "FV effective potential" $V_{\mathcal{N}=1}$ (given by (246)), the $\mathcal{N}=1$, $d=4$ holomorphic superpotential W, the *supersymmetry order parameters* $D_i Z = e^{\frac{1}{2}K} D_i W$ and the *axino-dilatino-CS modulino mixings* $D_0 D_i Z = e^{\frac{1}{2}K}\, D_0 D_i W$, which is part of the $(h_{2,1}+1) \times (h_{2,1}+1)$ *modulino mass matrix* $\Lambda_{ab} \equiv D_a D_b Z = e^{\frac{1}{2}K} D_a D_b W$ (note that in the considered $\mathcal{N}=1$, $d=4$ framework the axino-dilatino and the $h_{2,1}$ CS modulinos play the role of the $n_V = h_{2,1}$ CS modulinos in the context of $\mathcal{N}=2$, $d=4$ supergravity from Type IIB on CY_3). It is worth pointing out that Λ_{ab} is part of the holomorphic/anti-holomorphic form of the $(2h_{2,1}+2) \times (2h_{2,1}+2)$ covariant Hessian of Z, which is nothing but the holomorphic/anti-holomorphic form of the scalar (axion-dilaton+CS moduli, in the stringy description as Type IIB on $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$) mass matrix.

The structure of the criticality conditions (248) and (249) and (250) suggests the classification of the critical points of $V_{\mathcal{N}=1}$ into two general classes:

(I) The *supersymmetric (SUSY)* critical points of $V_{\mathcal{N}=1}$, determined by the differential constraints

$$D_a W = 0, \forall a = 0, 1, \ldots, h_{2,1}, \tag{251}$$

which directly solve the conditions (248) and (249) and (250). By substituting the SUSY FV constraints (251) into the expression (246) of $V_{\mathcal{N}=1}$, one obtains that SUSY dS critical points of $V_{\mathcal{N}=1}$ (i.e., independently on the stability, SUSY dS FV described by a classical FV Attractor Mechanism encoded by – the criticality conditions of – the potential $V_{\mathcal{N}=1}$) cannot exist, because

$$V_{\mathcal{N}=1,SUSY} = -3 \left(e^K |W|^2 \right)_{SUSY} = -3 |Z|^2_{SUSY} \leqslant 0. \tag{252}$$

(II) The *non-supersymmetric (non-SUSY)* critical points of $V_{\mathcal{N}=1}$, determined by the differential constraints

$$\begin{cases} D_a W \neq 0, \ (\textit{at least}) \text{ for some } a \in \{0,1,\ldots,h_{2,1}\}; \\ \\ \partial_a V_{\mathcal{N}=1} = 0, \forall a = 0,1,\ldots,h_{2,1}. \end{cases} \tag{253}$$

The expression (246) of $V_{\mathcal{N}=1}$ suggests that *a priori* such critical points of $V_{\mathcal{N}=1}$ are of all possible species (dS, Minkowski, AdS).

4.3 Supersymmetric Flux Vacua Attractor Equations

In the present subsection, we will concentrate on the SUSY critical points of $V_{\mathcal{N}=1}$, determining the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$. This can be achieved respectively by evaluating the Hodge identities (244), (245) and (285), (286), (287), (288) at the SUSY FV constraints (251).

The evaluation of the identities (244), (245) and (285), (286), (287), (288) along the constraints (251), respectively, yields the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in local "flat" coordinates[19]:

$$\begin{aligned} \mathfrak{F}_4 &= 2Re\left[\overline{Z}\hat{\Omega}_4 + \delta^{A\overline{B}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right]_{SUSY} \\ &= 2Re\left[\overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 + \delta^{I\overline{J}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{J}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_I\hat{\Omega}_3\right]_{SUSY} \\ &= 2e^{K_1+K_3}Re\left[\overline{W}\Omega_1\wedge\Omega_3 + \delta^{I\overline{J}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{J}}\overline{W}\right)\overline{\Omega}_1\wedge D_I\Omega_3\right]_{SUSY}. \end{aligned} \tag{254}$$

Notice that, as in (254), as well as in the treatment below, the subscript "*SUSY*" denotes the evaluation at the SUSY FV constraints (251).

Furthermore, (254) can be further elaborated by computing that

$$\begin{aligned} \left(D_{\underline{0}}D_J W\right)_{SUSY} &= \left(e^j_J\partial_j D_{\underline{0}}W\right)_{SUSY} = \left(e^j_J e^0_{\underline{0}}\partial_j D_0 W\right)_{SUSY} = \left(e^j_J e^0_{\underline{0}}\partial_0 D_j W\right)_{SUSY} \\ &= \left\{e^j_J\left(\overline{\tau}-\tau\right)\left[\partial_j\partial_0 W + \frac{1}{2}\left(\partial_j K_3\right)\partial_0 W\right]\right\}_{SUSY} \\ &= \left\{e^j_J\left(\overline{\tau}-\tau\right)\left[-q_{h|\Lambda}\partial_j X^\Lambda + p^\Lambda_h\partial_j F_\Lambda \right.\right. \\ &\quad \left.\left. + \frac{1}{2}\frac{\left(\overline{X}^\Sigma\partial_j F_\Sigma - \overline{F}_\Sigma\partial_j X^\Sigma\right)}{\overline{X}^\Delta F_\Delta - \overline{F}_\Delta X^\Delta}\left(q_{h|\Xi}X^\Xi - p^\Xi_h F_\Xi\right)\right]\right\}_{SUSY}. \end{aligned} \tag{255}$$

[19] For the analogous expression in generic local "curved" coordinates in M, see (289).

The structure of the SUSY FV AEs (254) suggests the classification of the SUSY critical points of $V_{\mathcal{N}=1}$ in three general classes:

(I) *Type "(3,0)"* SUSY FV, determined by the constraints (251) and by the further conditions

$$\begin{cases} W_{SUSY} \neq 0; \\ \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) \overline{\Omega}_1 \wedge D_i \Omega_3 \right]_{SUSY} = 0; \\ \Updownarrow \\ \forall \Lambda = 1, \dots, h_{2,1}+1 : \begin{cases} \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) D_i X^{\Lambda} \right]_{SUSY} = 0; \\ \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) D_i F_{\Lambda} \right]_{SUSY} = 0. \end{cases} \end{cases} \tag{256}$$

Because of

$$V_{\mathcal{N}=1,SUSY,(3,0)} = -3 \left(e^{K} |W|^2 \right)_{SUSY,(3,0)} < 0, \tag{257}$$

the class *"(3,0)"* of SUSY FV is composed only by AdS FV. In this case, the SUSY FV AEs read as follows:

$$\begin{aligned} \mathfrak{F}_4 &= 2Re \left[\overline{Z} \hat{\Omega}_4 \right]_{SUSY,(3,0)} = \\ &= 2Re \left[\overline{Z} \hat{\Omega}_1 \wedge \hat{\Omega}_3 \right]_{SUSY,(3,0)} = 2 \left(e^{K_1+K_3} \right)_{SUSY,(3,0)} Re \left[\overline{W} \Omega_1 \wedge \Omega_3 \right]_{SUSY,(3,0)}; \end{aligned} \tag{258}$$

Notice that for such a class of SUSY FV the condition of consistence of (258) is $W_{SUSY} \neq 0$. In other words, Minkowski $(V_{\mathcal{N}=1} = 0)$ SUSY FV satisfying the constraints (251) and

$$\begin{cases} W_{SUSY} = 0; \\ \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) \overline{\Omega}_1 \wedge D_i \Omega_3 \right]_{SUSY} = 0; \\ \Updownarrow \\ \forall \Lambda = 1, \dots, h_{2,1}+1 : \begin{cases} \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) D_i X^{\Lambda} \right]_{SUSY} = 0; \\ \left[g^{i\bar{j}} \left(\overline{D}_{\overline{0}} \overline{D}_{\bar{j}} \overline{W} \right) D_i F_{\Lambda} \right]_{SUSY} = 0. \end{cases} \end{cases} \tag{259}$$

are not described by the classical FV Attractor Mechanism encoded by the SUSY FV AEs (254). Indeed, such Equations, when evaluated along the constraints (259) simply return *all* (RR and NSNS) vanishing fluxes.

It is worth pointing out that, beside the substitution of "*Im*" with "*Re*" and the doubling of the vector dimension due to the $SL(2,\mathbb{R})$-doublet of RR and NSNS (3-form) fluxes, the *"(3,0)"* SUSY FV AEs (258) are very close to the SUSY extremal BH AEs in $\mathcal{N} = 2$, $d = 4$ supergravity, given by (154) and (155).

(II) *Type "(2,1)"* SUSY FV, determined by the constraints (251) and by the further conditions

$$\begin{cases} W_{SUSY}=0; \\ \qquad\qquad \left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3\right]_{SUSY}\neq 0; \\ \qquad\qquad \Updownarrow \\ (\textit{at least}) \text{ for some } \Lambda\in\{1,\dots,h_{2,1}+1\}: \begin{cases} \left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{W}\right)D_iX^{\Lambda}\right]_{SUSY}\neq 0; \\ \text{and/or} \\ \left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{W}\right)D_iF_{\Lambda}\right]_{SUSY}\neq 0. \end{cases} \end{cases} \tag{260}$$

Because of

$$V_{\mathscr{N}=1,SUSY,(2,1)}=-3\left(e^K|W|^2\right)_{SUSY,(2,1)}=0, \tag{261}$$

the class *"(2,1)"* of SUSY FV is composed only by Minkowski FV. In this case, the SUSY FV AEs read as follows:

$$\begin{aligned} \mathfrak{F}_4 &= \left[\delta^{A\overline{B}}\left(D_{\underline{0}}D_AZ\right)\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4+\delta^{B\overline{A}}\left(\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{A}}\overline{Z}\right)D_{\underline{0}}D_B\hat{\Omega}_4\right]_{SUSY,(2,1)} \\ &= 2Re\left[\left(t^0-\overline{t}^{\overline{0}}\right)g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3\right]_{SUSY,(2,1)} \\ &= 2Re\left[\overline{e}_{\underline{\overline{0}}}^{\overline{0}}e_I^i\overline{e}_{\overline{J}}^{\bar{j}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3\right]_{SUSY,(2,1)} \\ &= 2\left(e^{K_1+K_3}\right)_{SUSY,(2,1)}Re\left[\overline{e}_{\underline{\overline{0}}}^{\overline{0}}e_I^i\overline{e}_{\overline{J}}^{\bar{j}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\bar{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3\right]_{SUSY,(2,1)}. \end{aligned} \tag{262}$$

It is worth observing that the SUSY FV constraints (251) and the condition $W_{SUSY,(2,1)}=0$ imply

$$\begin{gathered} (\partial_aW)_{SUSY,(2,1)}=0,\forall a=0,1,\dots,h_{2,1}; \\ \Updownarrow \\ \begin{cases} (\partial_0W)_{SUSY,(2,1)}=\left(-q_{h|\Lambda}X^{\Lambda}+p_h^{\Lambda}F_{\Lambda}\right)_{SUSY,(2,1)}=0; \\ \forall i=1,\dots,h_{2,1}: \begin{cases} (\partial_iW)_{SUSY,(2,1)}=\left[q_{f|\Lambda}\partial_iX^{\Lambda}-p_f^{\Lambda}\partial_iF_{\Lambda}-\tau\left(q_{h|\Lambda}\partial_iX^{\Lambda}-p_h^{\Lambda}\partial_iF_{\Lambda}\right)\right]_{SUSY,(2,1)}=0; \\ \Updownarrow \\ \left(q_{h|\Lambda}\partial_iX^{\Lambda}-p_h^{\Lambda}\partial_iF_{\Lambda}\right)_{SUSY,(2,1)}=\left[\frac{1}{\tau}\left(q_{f|\Lambda}\partial_iX^{\Lambda}-p_f^{\Lambda}\partial_iF_{\Lambda}\right)\right]_{SUSY,(2,1)}, \end{cases} \end{cases} \end{gathered} \tag{263}$$

where we used the fact that $\tau\neq 0$ is a necessary (but not sufficient) condition for the (assumed) regularity of K_1 in $\mathscr{M}_{t^0\equiv\tau}\subsetneq M$ (or at least in the considered critical points of $V_{\mathscr{N}=1}$). Thus, (255) can be further elaborated as follows:

$$
\begin{aligned}
&\left(D_{\underline{0}}D_J W\right)_{SUSY,(2,1)} = \left(e_J^j \partial_j D_{\underline{0}} W\right)_{SUSY,(2,1)} \\
&= \left(e_J^j e_{\underline{0}}^0 \partial_j \partial_0 W\right)_{SUSY,(2,1)} = \left(e_J^j e_{\underline{0}}^0 \partial_0 \partial_j W\right)_{SUSY,(2,1)} \\
&= \left[e_J^j \left(\overline{\tau}-\tau\right) \partial_j \partial_0 W\right]_{SUSY,(2,1)} = \left\{e_J^j \left(\overline{\tau}-\tau\right)\left(-q_{h|\Lambda}\partial_j X^\Lambda + p_h^\Lambda \partial_j F_\Lambda\right)\right\}_{SUSY,(2,1)} \\
&= \left\{-e_J^j \frac{\left(\overline{\tau}-\tau\right)}{\tau}\left(q_{f|\Lambda}\partial_j X^\Lambda - p_f^\Lambda \partial_j F_\Lambda\right)\right\}_{SUSY,(2,1)},
\end{aligned}
\tag{264}
$$

where in the last line we used (263). Furthermore, by using (4.3.10) with some elaborations, one obtains that

$$
\left.\begin{array}{l} W_{SUSY,(2,1)} = 0 \\ \left(\partial_0 W\right)_{SUSY,(2,1)} = 0 \end{array}\right\} \Rightarrow \left(q_{f|\Lambda}X^\Lambda - p_f^\Lambda F_\Lambda\right)_{SUSY,(2,1)} = 0, \tag{265}
$$

and therefore at the class *"(2,1)"* of SUSY critical points of $V_{\mathcal{N}=1}$ the *"RR sector"* $q_{f|\Lambda}X^\Lambda - p_f^\Lambda F_\Lambda$ and the "*NSNS sector"* – $\left(q_{h|\Lambda}X^\Lambda - p_h^\Lambda F_\Lambda\right)$ of the holomorphic superpotential W vanish separately.

By looking at the *"(2,1)"* SUSY FV AEs (262), it is interesting to note that *"(2,1)"* SUSY FV do not have a counterpart in the theory of extremal BH attractors in $\mathcal{N}=2,\ d=4$ supergravity. Indeed, as implied by the SUSY extremal BH AEs (154) and (155), the classical extremal BH Attractor Mechanism in $\mathcal{N}=2,\ d=4$ supergravity is not consistent with SUSY critical points of V_{BH} also having $W=0$, and thus determining $V_{BH}=0$. In such a case, the SUSY extremal BH AEs (154) and (155) simply yield *all* (magnetic and electric) BH charges vanishing.

Contrarily to the extremal BH attractors in $\mathcal{N}=2,\ d=4$ supergravity, and as yielded by the *"(2,1)"* SUSY FV AEs (262), the classical FV Attractor Mechanism allows for stabilization of (axion-dilaton+CS) moduli in the SUSY case with vanishing *gravitino mass* $Z_{USY} = \left(e^K W\right)_{SUSY} = 0$.

(III) *Type* "$(3,0)+(2,1)$" SUSY FV, determined by the constraints (262) and by the further conditions

$$
\left\{
\begin{array}{l}
W_{SUSY} \neq 0; \\[2ex]
\qquad\qquad \left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1 \wedge D_i\Omega_3\right]_{SUSY} \neq 0; \\
\qquad\qquad\qquad \Updownarrow \\
(\textit{at least}) \text{ for some } \Lambda \in \{1,\ldots,h_{2,1}+1\}: \left\{
\begin{array}{l}
\left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_i X^\Lambda\right]_{SUSY} \neq 0; \\
\text{and/or} \\
\left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_i F_\Lambda\right]_{SUSY} \neq 0.
\end{array}\right.
\end{array}\right.
\tag{266}
$$

Because of

$$V_{\mathcal{N}=1,SUSY,(3,0)+(2,1)} = -3\left(e^{K}|W|^{2}\right)_{SUSY,(3,0)+(2,1)} < 0, \qquad (267)$$

the class "$(3,0)+(2,1)$" of SUSY FV is composed only by AdS FV. For such a class of SUSY FV the FV AEs are simply given by (254) (which can be further elaborated by considering (255)), constrained by (266).

In [11] examples of SUSY FV of all the classes considered above (*"(3,0)"*, *"(2,1)"*, and "$(3,0)+(2,1)$)" have been explicitly checked to satisfy the corresponding SUSY FV AEs of $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T_2}{\mathbb{Z}_2}$, in a model with $h_{2,1}=1$, where CY_3 is the so-called Fermat *sixtic* hypersurface.

5 Some Recent Developments on Extremal Black Hole Attractors

In these lectures, we have described the general theory of attractors for a generic $\mathcal{N}=2$, $d=4$ SK geometry, both in the supergravity language and in terms of Type IIB superstrings compactified on Calabi-Yau threefolds. We have then described, in a similar way, the Attractor Mechanism arising in $\mathcal{N}=1$, $d=4$ flux vacua, focussing on the case of (the F-theory limit of) compactifications of Type IIB on Calabi-Yau orientifolds (see also [123] for an extension to the landscape of non-Kähler vacua emerging in the flux compactifications of heterotic superstrings).

In the last years, more results have been obtained for the non-BPS extremal $d=4$ BH attractors, especially with regard to symmetric $\mathcal{N}=2$ SK geometries and to $\mathcal{N}>2$ extended theories.

The classification of the charge orbits of the U-duality [124] groups supporting attractors with non-vanishing entropy was performed in [76] (see also [19]) and [21], respectively for $\mathcal{N}=8$ and $\mathcal{N}=2$ symmetric supergravities, whereas the corresponding moduli spaces were found and studied, respectively, in [36] and [33]. Furthermore, the classification of attractors for $\mathcal{N}=3,\ 4$ (along with the corresponding maximal compact symmetries) was performed in [80]. Notice that the $\mathcal{N}=6$ theory has the same attractors, orbits and related moduli spaces of the quaternionic magic $\mathcal{N}=2$ model [21, 125].

For the sake of completeness we report here the charge orbits and the moduli space[20] of attractors for all $3\leq\mathcal{N}\leq 8$, $d=4$ supergravities (for the treatment of extremal BHs in such theories, see e.g. [117, 126, 127, 128, 129, 130]), including the cases $\mathcal{N}=3,\ 4,\ 5$, not exhaustively discussed in literature.

All $d=4$ theories with $\mathcal{N}$ even can be uplifted to $d=5$, and their U-duality group admits a unique quartic invariant (see e.g. [131]). All such supergravities have a non-BPS attractor solution whose moduli space coincide with the $d=5$ real

[20] The scalar manifolds of $3\leq\mathcal{N}\leq 8$, $d=4$ supergravities can be found e.g. in [80].

scalar manifold. This is the non-BPS solution with non-vanishing central charge matrix Z_{AB} $(A,\ B=1,\ldots,\mathcal{N})$, which breaks the $d=4$ $\mathcal{R}$-symmetry to the $d=5$ $\mathcal{R}$-symmetry. Since the cases $\mathcal{N}=6,\ 8$ have been treated in [19, 21, 33, 36, 80], let us now consider the case $\mathcal{N}=4$; as previously mentioned, its attractors with non-vanishing entropy (and the corresponding maximal compact symmetries) have been classified in [80]. The non-BPS attractor with $Z_{AB}\neq 0$ breaks the $d=4$ $\mathcal{R}$-symmetry $SU(4)\sim SO(6)$ down to the $d=5$ $\mathcal{R}$-symmetry $USp(4)\sim SO(5)$, and the maximal compact symmetry exhibited by the solution is $USp(4)\otimes SO(n-1)$, where n denotes the number of matter multiplets coupled to the supergravity one. The other non-BPS attractor solution of $\mathcal{N}=4,\ d=4$ supergravity has $Z_{AB}=0$; thus, the $d=4$ $\mathcal{R}$-symmetry $SU(4)$ is unbroken, and the corresponding maximal compact symmetry is $SU(4)\otimes SO(n-2)$.

On the other hand, the $d=4$ theories with $\mathcal{N}$ odd $(=3,\ 5)$ cannot be uplifted to $d=5$, and their U-duality group admits a unique quadratic invariant (see e.g. [131]). The case $\mathcal{N}=3$ admits only $\frac{1}{3}$-BPS and non-BPS $Z_{AB}=0$ attractors with non-vanishing entropy; notice that such a result is similar to the one obtained for the $\mathcal{N}=2$ symmetric sequence of SK manifolds based on quadratic holomorphic prepotential (see [21], [36] and Refs. therein), and it is ultimately due to the aforementioned fact that the $\mathcal{N}=3,\ d=4$ U-duality group $SU(3,n)$ has a unique quadratic (rather than quartic) invariant. $\mathcal{N}=5$ is peculiar, as discussed in [80], in such a case only the $\frac{1}{5}$-BPS attractor has non-vanishing entropy (this solution splits in BPS and non-BPS $Z=0$ ones when performing the $\mathcal{N}=5\rightarrow\mathcal{N}=2$ truncation of the theory [80]).

By knowing the real, symplectic representation R (with $dim_{\mathbb{R}}=\mathfrak{r}$) of the U-duality group G in which the charge vector Q sits, the orbits of R supporting attractors with non-vanishing entropy can be computed; their dimension is always $\mathfrak{r}-1$, because they are defined by a fixed, non-vanishing value of the unique U-invariant of the theory. For ($\mathcal{N}=2$ symmetric and) $3\leq\mathcal{N}\leq 8,\ d=4$ supergravities such orbits are homogeneous symmetric manifolds of the form $\frac{G}{\mathfrak{H}}$ ($\mathfrak{H}=\mathcal{H},\widehat{\mathcal{H}},\widetilde{\mathcal{H}}$, respectively, for BPS, non-BPS $Z_{AB}\neq 0$ and non-BPS $Z_{AB}=0$); the corresponding moduli space is given by the symmetric manifold $\frac{\mathfrak{H}}{h}$, where h $(=\mathfrak{h},\widehat{\mathfrak{h}},\widetilde{\mathfrak{h}}$, respectively) is the maximal compact subgroup of $\mathfrak{H}$. It is worth remarking that the $(\frac{1}{\mathcal{N}}-)$BPS moduli spaces of $3\leq\mathcal{N}\leq 8,\ d=4$ supergravities all are quaternionic Kähler manifolds; such a geometrical property can be understood by noticing that in the supersymmetry reduction down to $\mathcal{N}=2$ such spaces are spanned by the hypermultiplets' scalar degrees of freedom [131, 33].

Following [131] and [80], the relation among the signs of the U-invariant I_2 (quadratic in charges) or I_4 (quartic in charges) of the considered supergravity and the various BH charge orbits is (for the $\mathcal{N}=8$ case see also [19, 76, 78, 132, 133, 134, 135]):

$$\mathcal{N}=3:\qquad\begin{cases}\dfrac{1}{3}-BPS: I_2>0;\\ non-BPS, Z_{AB}=0: I_2<0;\end{cases}$$

$$
\begin{aligned}
\mathcal{N}=4: &\quad \begin{cases} \frac{1}{4}-BPS: I_4>0; \\ non-BPS, Z_{AB}\neq 0: I_4<0; \\ non-BPS, Z_{AB}=0: I_4>0; \end{cases} \\
\mathcal{N}=5: &\quad \frac{1}{5}-BPS: I_2 \gtrless 0 (sign\ does\ not\ matter); \\
\mathcal{N}=6: &\quad \begin{cases} \frac{1}{6}-BPS: I_4>0; \\ non-BPS, Z_{AB}\neq 0: I_4<0; \\ non-BPS, Z_{AB}=0: I_4>0; \end{cases} \\
\mathcal{N}=8: &\quad \begin{cases} \frac{1}{8}-BPS: I_4>0; \\ non-BPS, Z_{AB}\neq 0: I_4<0. \end{cases}
\end{aligned}
\tag{268}
$$

In Tables 1 and 2 we, respectively, list all charge orbits supporting extremal BH attractors with non-vanishing classical Bekenstein-Hawking [69, 70, 71, 72, 73] entropy (i.e. corresponding to the so-called "large" BHs) and their corresponding moduli spaces for $3 \leq \mathcal{N} \leq 8$, $d=4$ supergravities.

Some of the above results hold also for a generic (non-symmetric nor eventually homogeneous) $\mathcal{N}=2$, $d=4$ SK geometry based on a cubic holomorphic prepotential (usually named SK d-geometry [104]). For instance, for any SK d-geometry of $\mathcal{N}=2$, $d=4$ supergravity coupled to n Abelian vector multiplets, the so-called $D0$-$D6$ BH charge configuration supports only non-BPS $Z \neq 0$ attractors, whose moduli space is $(n-1)$-dimensional, and it is given by the corresponding $\mathcal{N}=2$, $d=5$ scalar manifold, endowed with real special geometry [34]. It is worth pointing out here that the existence of $n-1$ massless modes of the $2n \times 2n$ (real

Table 1 Charge orbits of the real, symplectic R representation of the U-duality group G supporting BH attractors with non-vanishing entropy in $3 \leq \mathcal{N} \leq 8$, $d=4$ supergravities

	$\frac{1}{\mathcal{N}}$-BPS orbits $\frac{G}{\mathcal{H}}$	Non-BPS, $Z_{AB}\neq 0$ orbits $\frac{G}{\widehat{\mathcal{H}}}$	Non-BPS, $Z_{AB}=0$ orbits $\frac{G}{\widetilde{\mathcal{H}}}$
$\mathcal{N}=3$	$\frac{SU(3,n)}{SU(2,n)}$	–	$\frac{SU(3,n)}{SU(3,n-1)}$
$\mathcal{N}=4$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(4,n)}$	$\frac{SU(1,1)}{SO(1,1)} \otimes \frac{SO(6,n)}{SO(5,n-1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6,n-2)}$
$\mathcal{N}=5$	$\frac{SU(1,5)}{SU(3)\otimes SU(2,1)}$	–	–
$\mathcal{N}=6$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$\mathcal{N}=8$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	–

Table 2 Moduli spaces of BH attractors with non-vanishing entropy in $3 \leq \mathcal{N} \leq 8$, $d = 4$ supergravities ($\mathfrak{h}$, $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$ are maximal compact subgroups of $\mathscr{H}$, $\widehat{\mathscr{H}}$ and $\widetilde{\mathscr{H}}$, respectively)

	$\frac{1}{\mathcal{N}}$-BPS moduli space $\frac{\mathscr{H}}{\mathfrak{h}}$	Non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\widehat{\mathscr{H}}}{\widehat{\mathfrak{h}}}$	Non-BPS, $Z_{AB} = 0$ moduli space $\frac{\widetilde{\mathscr{H}}}{\widetilde{\mathfrak{h}}}$
$\mathcal{N} = 3$	$\frac{SU(2,n)}{SU(2) \otimes SU(n) \otimes U(1)}$	–	$\frac{SU(3,n-1)}{SU(3) \otimes SU(n-1) \otimes U(1)}$
$\mathcal{N} = 4$	$\frac{SO(4,n)}{SO(4) \otimes SO(n)}$	$SO(1,1) \otimes \frac{SO(5,n-1)}{SO(5) \otimes SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6) \otimes SO(n-2)}$
$\mathcal{N} = 5$	$\frac{SU(2,1)}{SU(2) \otimes U(1)}$	–	–
$\mathcal{N} = 6$	$\frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}$	$\frac{SU^*(6)}{USp(6)}$	–
$\mathcal{N} = 8$	$\frac{E_{6(2)}}{SU(6) \otimes SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	–

from of the) Hessian matrix of the BH effective potential V_{BH} at its non-BPS $Z \neq 0$ critical points was shown in [10] to hold in any SK d-geometry of $\mathcal{N} = 2$, $d = 4$ supergravity coupled to n Abelian vector multiplets. However, the issue of the stability of the non-BPS $Z \neq 0$ critical points of V_{BH} (as well as of the non-BPS $Z = 0$ ones) in non-homogeneous SK d-geometry has not been thoroughly investigated so far[21].

Let us finally remark that it is also possible to relate the *flat* directions of non-BPS attractor solutions in $\mathcal{N} = 2$, $d = 4$ symmetric supergravities with the *flat* directions of ($\frac{1}{\mathcal{N}}$-)BPS and non-BPS attractors in $\mathcal{N} > 2$, $d = 4$ theories [33]. Moreover, the moduli spaces of extremal BH attractors with non-vanishing entropy in supergravity theories in $d = 5$ and $d = 6$ have been found and their relations with the corresponding Attractor Equation in $d = 4$ studied in [40] and [49].

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[21] The case of homogeneous non-symmetric SK d-geometry has been studied in [28].

Appendix I

Up to the third order of covariant differentiation included, the possible independent 4-forms ($(1,-1)$-Kähler weighted with respect to K) on CY_4 beside $\hat{\Omega}_4$ are:

$D_a\hat{\Omega}_4$:

$$\begin{cases} a=0: \\ D_0\hat{\Omega}_4 = \left(D_0\hat{\Omega}_1\right)\wedge\hat{\Omega}_3 = ie^{K_1}\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3 = \left(\bar{t}^{\bar{0}}-t^0\right)^{-1}\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3; \\ a=i: \\ D_i\hat{\Omega}_4 = \hat{\Omega}_1\wedge D_i\hat{\Omega}_3 = e^{\frac{1}{2}K_3}\hat{\Omega}_1\wedge D_i\Omega_3 = e^{\frac{1}{2}K_3}\hat{\Omega}_1\wedge\left[\partial_i\Omega_3+(\partial_iK_3)\,\Omega_3\right] \\ = \dfrac{1}{\sqrt{i\left(\overline{X}^{\Delta}F_{\Delta}-X^{\Delta}\overline{F}_{\Delta}\right)}}\hat{\Omega}_1\wedge\left\{\begin{array}{l}\left[\partial_iX^{\Lambda}-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}X^{\Lambda}\right]\alpha_{\Lambda}+ \\ -\left[\partial_iF_{\Lambda}-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}F_{\Lambda}\right]\beta^{\Lambda}\end{array}\right\}. \end{cases} \tag{269}$$

$D_aD_b\hat{\Omega}_4 = D_{(a}D_{b)}\hat{\Omega}_4$:

$$\begin{cases} (a,b)=(0,0): D_0D_0\hat{\Omega}_4 = 0; \\ (a,b)=(0,i): \\ D_0D_i\hat{\Omega}_4 = D_0\hat{\Omega}_1\wedge D_i\hat{\Omega}_3 \\ = \dfrac{1}{\left(\bar{t}^{\bar{0}}-t^0\right)\sqrt{i\left(\overline{X}^{\Delta}F_{\Delta}-X^{\Delta}\overline{F}_{\Delta}\right)}}\overline{\hat{\Omega}}_1\wedge\left\{\begin{array}{l}\left[\partial_iX^{\Lambda}-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}X^{\Lambda}\right]\alpha_{\Lambda}+ \\ -\left[\partial_iF_{\Lambda}-\dfrac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}F_{\Lambda}\right]\beta^{\Lambda}\end{array}\right\}; \\ (a,b)=(i,j): \\ D_iD_j\hat{\Omega}_4 = \hat{\Omega}_1\wedge D_iD_j\hat{\Omega}_3 = iC_{ijk}g^{k\bar{l}}\hat{\Omega}_1\wedge\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_3 \\ = -e^{-K_1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_1\wedge\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_3 = -e^{-K_1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{0}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_4 \\ = -e^{-K_1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4 = -i\left(\bar{t}^{\bar{0}}-t^0\right)C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4. \end{cases} \tag{270}$$

$$D_aD_bD_c\hat{\Omega}_4 = D_{(a}D_bD_{c)}\hat{\Omega}_4 :$$

$$\begin{cases} (a,b,c) = (0,0,0) : D_0D_0D_0\hat{\Omega}_4 = 0; \\ (a,b,c) = (0,0,i) : D_0D_0D_i\hat{\Omega}_4 = 0; \\ (a,b,c) = (0,i,j) : \begin{cases} D_0D_iD_j\hat{\Omega}_4 = D_0\hat{\Omega}_1 \wedge D_iD_j\hat{\Omega}_3 \\ = -e^{K_1}C_{ijk}g^{k\bar{l}}\overline{\hat{\Omega}}_1 \wedge \overline{D}_{\bar{l}}\overline{\hat{\Omega}}_3 \\ = -e^{K_1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_4 = i\left(\bar{t}^{\bar{0}} - t^0\right)^{-1} C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_4; \end{cases} \\ (a,b,c) = (i,j,k) : \begin{cases} D_iD_jD_k\hat{\Omega}_4 = -i\left(\bar{t}^{\bar{0}} - t^0\right)\left(D_iC_{jkl}\right)g^{l\overline{m}}\overline{D}_{\overline{m}}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4 + \\ -i\left(\bar{t}^{\bar{0}} - t^0\right)C_{jkl}g^{l\overline{m}}D_i\overline{D}_{\overline{m}}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4 \\ = -i\left(\bar{t}^{\bar{0}} - t^0\right)\left(D_iC_{jkl}\right)g^{l\overline{m}}\overline{D}_{\bar{0}}\overline{D}_{\overline{m}}\overline{\hat{\Omega}}_4 + \\ -i\left(\bar{t}^{\bar{0}} - t^0\right)C_{jkl}g^{l\overline{m}}\left(\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_1\right) \wedge D_i\overline{D}_{\overline{m}}\overline{\hat{\Omega}}_3 \\ = -i\left(\bar{t}^{\bar{0}} - t^0\right)\left(D_iC_{jkl}\right)g^{l\overline{m}}\overline{D}_{\bar{0}}\overline{D}_{\overline{m}}\overline{\hat{\Omega}}_4 - i\left(\bar{t}^{\bar{0}} - t^0\right)C_{ijk}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4. \end{cases} \end{cases} \tag{271}$$

$$\overline{D}_{\bar{a}}D_bD_c\hat{\Omega}_4 = \overline{D}_{\bar{a}}D_{(b}D_{c)}\hat{\Omega}_4 :$$

$$\begin{cases} (\bar{a},b,c) = \left(\bar{0},0,0\right) : \overline{D}_{\bar{0}}D_0D_0\hat{\Omega}_4 = 0; \\ (\bar{a},b,c) = \left(\bar{0},0,i\right) : \overline{D}_{\bar{0}}D_0D_i\hat{\Omega}_4 = g_{0\bar{0}}D_i\hat{\Omega}_4 = -\left(\bar{t}^{\bar{0}} - t^0\right)^{-2}D_i\hat{\Omega}_4 = e^{2K_1}D_i\hat{\Omega}_4; \\ (\bar{a},b,c) = \left(\bar{0},i,j\right) : \overline{D}_{\bar{0}}D_iD_j\hat{\Omega}_4 = 0; \\ (\bar{a},b,c) = \left(\bar{l},0,0\right) : \overline{D}_{\bar{l}}D_0D_0\hat{\Omega}_4 = 0; \\ (\bar{a},b,c) = \left(\bar{l},0,i\right) : \overline{D}_{\bar{l}}D_0D_i\hat{\Omega}_4 = g_{i\bar{l}}D_0\hat{\Omega}_4; \\ (\bar{a},b,c) = \left(\bar{l},i,j\right) : \\ \overline{D}_{\bar{l}}D_iD_j\hat{\Omega}_4 = iC_{ijk}g^{k\bar{k}}\hat{\Omega}_1 \wedge \overline{D}_{\bar{l}}\overline{D}_{\bar{k}}\overline{\hat{\Omega}}_3 \\ = g^{k\bar{k}}C_{ijk}\overline{C}_{\bar{l}\overline{m}\bar{k}}g^{m\overline{m}}\hat{\Omega}_1 \wedge D_m\hat{\Omega}_3 \\ = g^{k\bar{k}}C_{ijk}\overline{C}_{\bar{l}\overline{m}\bar{k}}g^{m\overline{m}}D_m\hat{\Omega}_4 \\ \overset{SKG\ constraints}{=} \left(R_{i\bar{l}j\overline{m}}g^{m\overline{m}} + \delta_j^m g_{i\bar{l}} + \delta_i^m g_{j\bar{l}}\right)D_m\hat{\Omega}_4. \end{cases} \tag{272}$$

Since the covariant derivatives of $\hat{\Omega}_4$ are often considered in local "flat" coordinated in M, below we write the independent ones, up to the third order included, by recalling (208) and (211) (implemented by (4.1.1)), (205) and (212), and (269), (270), (271), (272):

$$D_A\hat{\Omega}_4: \begin{cases} A=\underline{0}: D_{\underline{0}}\hat{\Omega}_4 = e^a_{\underline{0}} D_a\hat{\Omega}_4 = e^0_{\underline{0}}\left(D_0\hat{\Omega}_1\right)\wedge\hat{\Omega}_3 = \overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3; \\ A=I: D_I\hat{\Omega}_4 = e^a_I D_a\hat{\Omega}_4 = e^i_I D_i\hat{\Omega}_4 = e^i_I\left(\hat{\Omega}_1\wedge D_i\hat{\Omega}_3\right). \end{cases} \tag{273}$$

$$D_AD_B\hat{\Omega}_4 = D_{(A}D_{B)}\hat{\Omega}_4:$$

$$\begin{cases} (A,B)=(\underline{0},\underline{0}): D_{\underline{0}}D_{\underline{0}}\hat{\Omega}_4 = e^a_{\underline{0}}e^b_{\underline{0}}D_aD_b\hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^2 D_0D_0\hat{\Omega}_4 = 0; \\ (A,B)=(\underline{0},I): D_{\underline{0}}D_I\hat{\Omega}_4 = e^a_{\underline{0}}e^b_I D_aD_b\hat{\Omega}_4 = e^0_{\underline{0}}e^i_I D_0D_i\hat{\Omega}_4 = D_{\underline{0}}\hat{\Omega}_1\wedge D_I\hat{\Omega}_3 = \overline{\hat{\Omega}}_1\wedge D_I\hat{\Omega}_3; \\ (A,B)=(I,J): \begin{cases} D_ID_J\hat{\Omega}_4 = e^a_Ie^b_JD_aD_b\hat{\Omega}_4 \\ = e^i_Ie^j_JD_iD_j\hat{\Omega}_4 = e^i_Ie^j_J\left(\hat{\Omega}_1\wedge D_iD_j\hat{\Omega}_3\right) \\ = -e^i_Ie^j_Je^{-K_1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{0}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_4 = iC_{IJK}\delta^{K\overline{K}}\hat{\Omega}_1\wedge\overline{D}_{\overline{K}}\overline{\hat{\Omega}}_3 \\ = iC_{IJK}\delta^{K\overline{K}}\overline{D}_{\underline{\bar{0}}}\overline{\hat{\Omega}}_1\wedge\overline{D}_{\overline{K}}\overline{\hat{\Omega}}_3. \end{cases} \end{cases} \tag{274}$$

$$D_AD_BD_C\hat{\Omega}_4 = D_{(A}D_BD_{C)}\hat{\Omega}_4:$$

$$\begin{cases} (A,B,C)=(\underline{0},\underline{0},\underline{0}): D_{\underline{0}}D_{\underline{0}}D_{\underline{0}}\hat{\Omega}_4 = e^a_{\underline{0}}e^b_{\underline{0}}e^c_{\underline{0}}D_aD_bD_c\hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^3 D_0D_0D_0\hat{\Omega}_4 = 0; \\ (A,B,C)=(\underline{0},\underline{0},I): D_{\underline{0}}D_{\underline{0}}D_I\hat{\Omega}_4 = e^a_{\underline{0}}e^b_{\underline{0}}e^c_ID_aD_bD_c\hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^2 e^i_ID_0D_0D_i\hat{\Omega}_4 = 0; \\ (A,B,C)=(\underline{0},I,J): \begin{cases} D_{\underline{0}}D_ID_J\hat{\Omega}_4 = e^a_{\underline{0}}e^b_Ie^c_JD_aD_bD_c\hat{\Omega}_4 = e^0_{\underline{0}}e^i_Ie^j_JD_0D_iD_j\hat{\Omega}_4 \\ = ie^0_{\underline{0}}e^i_Ie^j_J\left(\bar{t}^{\bar{0}}-t^0\right)^{-1}C_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{\hat{\Omega}}_4 = iC_{IJK}\delta^{K\overline{K}}\overline{D}_{\overline{K}}\overline{\hat{\Omega}}_4; \end{cases} \\ (A,B,C)=(I,J,K): \begin{cases} D_ID_JD_K\hat{\Omega}_4 = e^a_Ie^b_Je^c_KD_aD_bD_c\hat{\Omega}_4 = e^i_Ie^j_Je^k_KD_iD_jD_k\hat{\Omega}_4 \\ = -ie^i_Ie^j_Je^k_K\left(\bar{t}^{\bar{0}}-t^0\right)\left(D_iC_{jkl}\right)g^{l\overline{m}}\overline{D}_{\bar{0}}\overline{D}_{\overline{m}}\overline{\hat{\Omega}}_4 + \\ -ie^i_Ie^j_Je^k_K\left(\bar{t}^{\bar{0}}-t^0\right)C_{ijk}\overline{D}_{\bar{0}}\overline{\hat{\Omega}}_4 \\ = i\left(D_IC_{JKL}\right)\delta^{L\overline{L}}\overline{D}_{\underline{\bar{0}}}\overline{D}_{\overline{L}}\overline{\hat{\Omega}}_4 + iC_{IJK}\overline{D}_{\underline{\bar{0}}}\overline{\hat{\Omega}}_4. \end{cases} \end{cases} \tag{275}$$

$\overline{D}_{\overline{A}} D_B D_C \hat{\Omega}_4 = \overline{D}_{\overline{A}} D_{(B} D_{C)} \hat{\Omega}_4$:

$$
\begin{cases}
\left(\overline{A}, B, C\right) = \left(\overline{\underline{0}}, \underline{0}, \underline{0}\right) : \overline{D}_{\overline{\underline{0}}} D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^b_{\underline{0}} e^c_{\underline{0}} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} \left(e^0_{\underline{0}}\right)^2 \overline{D}_{\overline{0}} D_0 D_0 \hat{\Omega}_4 = 0; \\
\left(\overline{A}, B, C\right) = \left(\overline{\underline{0}}, \underline{0}, I\right) : \overline{D}_{\overline{\underline{0}}} D_{\underline{0}} D_I \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^b_{\underline{0}} e^c_I \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 \\
= \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^0_{\underline{0}} e^i_I \overline{D}_{\overline{0}} D_0 D_i \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^0_{\underline{0}} e^i_I g_{0\overline{0}} D_i \hat{\Omega}_4 = D_I \hat{\Omega}_4; \\
\left(\overline{A}, B, C\right) = \left(\overline{\underline{0}}, I, J\right) : \overline{D}_{\overline{\underline{0}}} D_I D_J \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^b_I e^c_J \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^i_I e^j_J \overline{D}_{\overline{0}} D_i D_j \hat{\Omega}_4 = 0; \\
\left(\overline{A}, B, C\right) = \left(\overline{L}, \underline{0}, \underline{0}\right) : \overline{D}_{\overline{L}} D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^b_{\underline{0}} e^c_{\underline{0}} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} \left(e^0_{\underline{0}}\right)^2 \overline{D}_{\overline{l}} D_0 D_0 \hat{\Omega}_4 = 0; \\
\left(\overline{A}, B, C\right) = \left(\overline{L}, \underline{0}, I\right) : \overline{D}_{\overline{L}} D_{\underline{0}} D_I \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^b_{\underline{0}} e^c_I \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 \\
= \overline{e}^{\overline{l}}_{\overline{L}} e^0_{\underline{0}} e^i_I \overline{D}_{\overline{l}} D_0 D_i \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} e^0_{\underline{0}} e^i_I g_{i\overline{l}} D_0 \hat{\Omega}_4 = \delta_{I\overline{L}} D_{\underline{0}} \hat{\Omega}_4; \\
\left(\overline{A}, B, C\right) = \left(\overline{L}, I, J\right) : \begin{cases}
\overline{D}_{\overline{L}} D_I D_J \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^b_I e^c_J \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} e^i_I e^j_J \overline{D}_{\overline{l}} D_i D_j \hat{\Omega}_4 \\
= \overline{e}^{\overline{l}}_{\overline{L}} e^i_I e^j_J g^{k\overline{k}} C_{ijk} \overline{C}_{\overline{l}\overline{m}\overline{k}} g^{m\overline{m}} D_m \hat{\Omega}_4 \\
\overset{\text{SKG constraints in local “flat” coords.}}{=} \left(\delta^M_J \delta_{I\overline{L}} + \delta^M_I \delta_{J\overline{L}}\right) D_M \hat{\Omega}_4.
\end{cases}
\end{cases}
\tag{276}
$$

Appendix II

The *“intersections”* among the elements of the set of 4-forms $\hat{\Omega}_4$, $D_0\hat{\Omega}_4$, $D_i\hat{\Omega}_4$, $D_0 D_i \hat{\Omega}_4$, $\overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{i}} \overline{\hat{\Omega}}_4$ and $\overline{D}_{\overline{0}} \overline{D}_{\overline{i}} \overline{\hat{\Omega}}_4$ in generic local “curved” and in local “flat” coordinates of M, respectively, read as follows:

$$
\begin{gathered}
\int_{CY_4} \hat{\Omega}_4 \wedge \hat{\Omega}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge D_0 \hat{\Omega}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge D_i \hat{\Omega}_4 = 0, \\
\int_{CY_4} \hat{\Omega}_4 \wedge D_0 D_i \hat{\Omega}_4 = 0; \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{\hat{\Omega}}_4 = 1; \\
\int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\overline{i}} \overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\overline{0}} \overline{D}_{\overline{i}} \overline{\hat{\Omega}}_4 = 0;
\end{gathered}
\tag{277}
$$

$$
\begin{gathered}
\int_{CY_4} D_i \hat{\Omega}_4 \wedge D_j \hat{\Omega}_4 = 0, \quad \int_{CY_4} D_i \hat{\Omega}_4 \wedge D_0 \hat{\Omega}_4 = 0, \quad \int_{CY_4} D_i \hat{\Omega}_4 \wedge D_0 D_j \hat{\Omega}_4 = 0; \\
\int_{CY_4} D_i \hat{\Omega}_4 \wedge \overline{D}_{\overline{j}} \overline{\hat{\Omega}}_4 = -g_{i\overline{j}}; \\
\int_{CY_4} D_i \hat{\Omega}_4 \wedge \overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} D_i \hat{\Omega}_4 \wedge \overline{D}_{\overline{0}} \overline{D}_{\overline{j}} \overline{\hat{\Omega}}_4 = 0;
\end{gathered}
\tag{278}
$$

$$\int_{CY_4} D_0\hat{\Omega}_4 \wedge D_0\hat{\Omega}_4 = 0, \quad \int_{CY_4} D_0\hat{\Omega}_4 \wedge D_0 D_i\hat{\Omega}_4 = 0;$$

$$\int_{CY_4} D_0\hat{\Omega}_4 \wedge \overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4 = -e^{2K_1} = \left(\overline{t}^{\overline{0}} - t^0\right)^{-2}; \tag{279}$$

$$\int_{CY_4} D_0\hat{\Omega}_4 \wedge \overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4 = 0;$$

$$\int_{CY_4} D_0 D_i\hat{\Omega}_4 \wedge D_0 D_j\hat{\Omega}_4 = 0;$$

$$\int_{CY_4} D_0 D_i\hat{\Omega}_4 \wedge \overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{\hat{\Omega}}_4 = e^{2K_1} g_{i\overline{j}} = -\left(\overline{t}^{\overline{0}} - t^0\right)^{-2} g_{i\overline{j}}. \tag{280}$$

$$\int_{CY_4} \hat{\Omega}_4 \wedge \hat{\Omega}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge D_{\underline{0}}\hat{\Omega}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge D_I\hat{\Omega}_4 = 0,$$

$$\int_{CY_4} \hat{\Omega}_4 \wedge D_{\underline{0}} D_I\hat{\Omega}_4 = 0; \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{\hat{\Omega}}_4 = 1; \tag{281}$$

$$\int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} \hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4 = 0;$$

$$\int_{CY_4} D_I\hat{\Omega}_4 \wedge D_J\hat{\Omega}_4 = 0, \quad \int_{CY_4} D_I\hat{\Omega}_4 \wedge D_{\underline{0}}\hat{\Omega}_4 = 0, \quad \int_{CY_4} D_I\hat{\Omega}_4 \wedge D_{\underline{0}} D_J\hat{\Omega}_4 = 0;$$

$$\int_{CY_4} D_I\hat{\Omega}_4 \wedge \overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4 = -e_I^i \overline{e}_{\overline{J}}^{\overline{j}} g_{i\overline{j}} = -\delta_{I\overline{J}}; \tag{282}$$

$$\int_{CY_4} D_I\hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4 = 0, \quad \int_{CY_4} D_I\hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4 = 0;$$

$$\int_{CY_4} D_{\underline{0}}\hat{\Omega}_4 \wedge D_{\underline{0}}\hat{\Omega}_4 = 0, \quad \int_{CY_4} D_{\underline{0}}\hat{\Omega}_4 \wedge D_{\underline{0}} D_I\hat{\Omega}_4 = 0;$$

$$\int_{CY_4} D_{\underline{0}}\hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4 = -\left|e_{\underline{0}}^0\right|^2 e^{2K_1} = -1; \tag{283}$$

$$\int_{CY_4} D_{\underline{0}}\hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4 = 0;$$

$$\int_{CY_4} D_{\underline{0}} D_I\hat{\Omega}_4 \wedge D_{\underline{0}} D_J\hat{\Omega}_4 = 0;$$

$$\int_{CY_4} D_{\underline{0}} D_I\hat{\Omega}_4 \wedge \overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4 = \left|e_{\underline{0}}^0\right|^2 e_I^i \overline{e}_{\overline{J}}^{\overline{j}} e^{2K_1} g_{i\overline{j}} = \delta_{I\overline{J}}. \tag{284}$$

Appendix III

The complete Hodge-decomposition of the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ of Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in generic local "curved" coordinates in M reads as follows:

$$\mathfrak{F}_4=\left[\begin{array}{l} Z\overline{\hat{\Omega}}_4-g^{a\bar{b}}\left(D_aZ\right)\overline{D}_{\bar{b}}\overline{\hat{\Omega}}_4+\left|e_{\underline{0}}^0\right|^2g^{a\bar{b}}\left(D_0D_aZ\right)\overline{D}_{\bar{0}}\overline{D}_{\bar{b}}\overline{\hat{\Omega}}_4+\\ \\ +\left|e_{\underline{0}}^0\right|^2g^{b\bar{a}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{a}}\overline{Z}\right)D_0D_b\hat{\Omega}_4-g^{b\bar{a}}\left(\overline{D}_{\bar{a}}\overline{Z}\right)D_b\hat{\Omega}_4+\overline{Z}\hat{\Omega}_4 \end{array}\right]$$

$$=2Re\left[\overline{Z}\hat{\Omega}_4-g^{a\bar{b}}\left(\overline{D}_{\bar{b}}\overline{Z}\right)D_a\hat{\Omega}_4+\left|e_{\underline{0}}^0\right|^2g^{a\bar{b}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{b}}\overline{Z}\right)D_0D_a\hat{\Omega}_4\right] \tag{285}$$

$$=2Re\left[\begin{array}{l} \overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3+\\ \\ +\left(t^0-\bar{t}^{\bar{0}}\right)^2\left(\overline{D}_{\bar{0}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3-g^{i\bar{j}}\left(\overline{D}_{\bar{j}}\overline{Z}\right)\hat{\Omega}_1\wedge D_i\hat{\Omega}_3+\\ \\ +\left(t^0-\bar{t}^{\bar{0}}\right)g^{i\bar{j}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3 \end{array}\right] \tag{286}$$

$$=2Re\left[\begin{array}{l} \overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3+\\ \\ -\left|e_{\underline{0}}^0\right|^2\left(\overline{D}_{\bar{0}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3-e_I^i\bar{e}_{\bar{J}}^{\bar{j}}\delta^{I\bar{J}}\left(\overline{D}_{\bar{j}}\overline{Z}\right)\hat{\Omega}_1\wedge D_i\hat{\Omega}_3+\\ \\ +\bar{e}_{\underline{\bar{0}}}^{\bar{0}}e_I^i\bar{e}_{\bar{J}}^{\bar{j}}\delta^{I\bar{J}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3 \end{array}\right] \tag{287}$$

$$=2e^{K_1+K_3}Re\left[\begin{array}{l} \overline{W}\Omega_1\wedge\Omega_3+\\ \\ -\left|e_{\underline{0}}^0\right|^2\left(\overline{D}_{\bar{0}}\overline{W}\right)\overline{\Omega}_1\wedge\Omega_3-e_I^i\bar{e}_{\bar{J}}^{\bar{j}}\delta^{I\bar{J}}\left(\overline{D}_{\bar{j}}\overline{W}\right)\Omega_1\wedge D_i\Omega_3+\\ \\ +\bar{e}_{\underline{\bar{0}}}^{\bar{0}}e_I^i\bar{e}_{\bar{J}}^{\bar{j}}\delta^{I\bar{J}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3 \end{array}\right]. \tag{288}$$

The evaluation of such identities along the constraints (251) yields the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in local "curved" coordinates:

$$\mathfrak{F}_4=\left[Z\overline{\hat{\Omega}}_4+\left|e_{\underline{0}}^0\right|^2g^{a\bar{b}}\left(D_0D_aZ\right)\overline{D}_{\bar{0}}\overline{D}_{\bar{b}}\overline{\hat{\Omega}}_4+\left|e_{\underline{0}}^0\right|^2g^{b\bar{a}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{a}}\overline{Z}\right)D_0D_b\hat{\Omega}_4+\overline{Z}\hat{\Omega}_4\right]_{SUSY}$$

$$=2Re\left[\overline{Z}\hat{\Omega}_4+\left|e_{\underline{0}}^0\right|^2g^{a\bar{b}}\left(\overline{D}_{\bar{0}}\overline{D}_{\bar{b}}\overline{Z}\right)D_0D_a\hat{\Omega}_4\right]_{SUSY}$$

$$= 2Re\left[\overline{Z}\hat{\Omega}_1 \wedge \hat{\Omega}_3 + \left(t^0 - \overline{t}^{\overline{0}}\right) g^{i\overline{j}} \left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right) \overline{\hat{\Omega}}_1 \wedge D_i\hat{\Omega}_3\right]_{SUSY}$$

$$= 2Re\left[\overline{Z}\hat{\Omega}_1 \wedge \hat{\Omega}_3 + \overline{e}_{\underline{0}}^{\overline{0}} e_I^i \overline{e}_{\overline{J}}^{\overline{j}} \delta^{I\overline{J}} \left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right) \overline{\hat{\Omega}}_1 \wedge D_i\hat{\Omega}_3\right]_{SUSY}$$

$$= 2e^{K_1+K_3} Re\left[\overline{W}\Omega_1 \wedge \Omega_3 + \overline{e}_{\underline{0}}^{\overline{0}} e_I^i \overline{e}_{\overline{J}}^{\overline{j}} \delta^{I\overline{J}} \left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right) \overline{\Omega}_1 \wedge D_i\Omega_3\right]_{SUSY}. \quad (289)$$

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Lectures on Black Holes and the AdS_3/CFT_2 Correspondence

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Abstract We present a detailed discussion of AdS_3 black holes and their connection to two-dimensional conformal field theories via the AdS/CFT correspondence. Our emphasis is on deriving refined versions of black hole partition functions that include the effect of higher derivative terms in the spacetime action as well as non-perturbative effects. We include background material on gravity in AdS_3, in the context of holographic renormalization.

1 Introduction

The fact that string theory is able to provide a successful microscopic description of certain black holes provides strong evidence that it is a consistent theory of quantum gravity. Correctly reproducing the Bekenstein-Hawking entropy formula $S = A/4G$ from an explicit sum over states indicates that the right microscopic degrees of freedom have been identified. Since string theory also reduces to conventional general relativity (coupled to matter) at low energy, it seems to provide us with a coherent theory encompassing both the microscopic and macroscopic regimes. Needless to say, however, there is still much to be learned about the full implications of string theory for quantum gravity.

One approach to deepening our understanding is to examine the string theory description of black holes with improved precision. This program has been highly fruitful so far. The earliest successful black hole entropy matches, following [1], appeared somewhat miraculous, the emergence of the Bekenstein-Hawking formula from the microscopic side not becoming apparent until all the last numerical factors were accounted for. The precise agreement seemed even more astonishing once additional features like rotation and non-extremality were included. It was eventually understood that the essential ingredients on the two sides are the near horizon AdS region of the black hole geometry, and the low energy CFT describing

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the underlying branes, and this (among other observations) led to the celebrated AdS/CFT correspondence [2, 3]. This improved understanding largely demystifies the nature of the entropy matching, as we will discuss in these lectures. One of our goals here will be to show how just a few basic features, like the existence of a near horizon AdS region with the appropriate symmetries, is enough to make the agreement manifest, even in rather complicated contexts, and including incorporating subleading corrections to the area law formula.

A survey of the examples in which there is a precise microscopic accounting of black entropy reveals the near ubiquitous appearance of a near horizon AdS_3 factor (possibly after a suitable duality transformation).[1] In these examples, the dual theory is a two-dimensional CFT, for which there are powerful results constraining the spectrum of states. By contrast, in other examples such as AdS_5 black holes, it has so far only been possible to compute the entropy up to at best numerical factors. For this reason, here we will be focussing on AdS_3 examples.

The AdS_3/CFT_2 correspondence can be stated as an equivalence between partition functions

$$Z_{AdS} = Z_{CFT}. \tag{1}$$

The connection with black hole entropy arises when we examine this relation in the high energy regime, where the left-hand side is dominated by an asymptotically AdS_3 black hole: the BTZ black hole [5, 6]. General properties of conformal field theories imply that the asymptotic density of states will agree between the two sides, as we will discuss in what follows.

A more ambitious goal is to try to demonstrate *exact* agreement in (1). On the gravitational side, this will involve incorporating many new contributions beyond that of a single large black hole. One way to think of defining Z_{AdS} is as a Euclidean path integral. At finite temperature, the contributing Euclidean geometries should have a boundary that is a two-dimensional torus, to match with the standard finite temperature description of the boundary CFT. A typical bulk geometry that is thereby included is one whose topology is a three-dimensional solid torus. Such a geometry appears in the path integral weighted by its Euclidean action, which includes (if we are trying to be exact) contributions from an infinite series of higher derivative terms in the spacetime Lagrangian. That is not all though, since we also need to include all possible excitations on top of the geometry, allowing for particle, string, and brane states that can wind around the solid torus. After all these contributions have been taken into account one can hope to match the exact CFT partition function.

In these notes, we will discuss to what extent this program can be carried out. This will involve a careful study of gravity in asymptotically AdS_3 spacetimes, and its string theory realization. We organize our presentation by starting with a fairly generic setup and then becoming progressively more specific. As we will see, once we start adding more structure, like supersymmetry, to the problem, and refine our

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[1] For a recent example without such an AdS_3 factor see [4]. But note that in this example, the microscopic counting is not under complete control and interestingly still involves relating the system to another system which *does* have an AdS_3 region.

definitions of the partition functions in (1), it is possible to go a significant distance in demonstrating exact agreement between the gravitational and CFT descriptions.

Let us describe the outcome of our analysis in a bit more detail. In a two-dimensional CFT we have independent temperatures for the left and right movers; we label the inverse left(right) moving temperature as $\tau \sim 1/T_L$ ($\overline{\tau} \sim 1/T_R$). Further, in the CFT there is a spectrum of left and right moving conserved charges, and we can turn on chemical potentials for these charges, z_I and $\tilde{z}_I$. Allowing for nonzero potentials lets us study charged black holes. To study black hole entropy, we are interested in the high temperature behavior of the partition function, and we will see that it has the structure

$$\begin{aligned} \ln Z = \frac{i\pi}{\tau}\left(\frac{c}{12} - 2C^{IJ} z_I z_J\right) - \frac{i\pi}{\overline{\tau}}\left(\frac{\tilde{c}}{12} - 2\tilde{C}^{IJ}\tilde{z}_i\tilde{z}_J\right) \\ + \text{exponentially suppressed terms.} \end{aligned} \tag{2}$$

Here c and $\tilde{c}$ are the left and right moving central charges, and C^{IJ} and $\tilde{C}^{IJ}$ are matrices appearing in the CFT current algebra. On the gravity side, if we use the two-derivative approximation to the spacetime action and discard the exponentially small terms, we will reproduce the area law for the entropy of a general rotating, charged black hole. But we can go considerably further: the parameters c, etc., can be computed *exactly* by relating them to anomalies. The corrections to these parameters encode the effect of higher derivative terms in the spacetime action, and lead to corrections to the area law. Indeed, by transforming (2) into an expression for the degeneracy as a function of charges (i.e. relating the canonical ensemble to the microcanonical ensemble via a Laplace transform) we deduce a series of $1/Q$ corrections to the degeneracy, as in [7, 8]. The suppressed terms in the second line of (2) will arise, in the gravitational description, from including fluctuations around black hole geometries, and from summing over inequivalent black holes.

As we proceed, it will become clear that terms on the first and second lines of (2) are of a rather different nature. The top line can be established on general grounds, using the relation to anomalies. In particular, this can be achieved even for non-BPS and nonextremal black hole, and so a class of area law corrections for such black holes are under excellent control.[2] Also, our method of derivation will make it manifest that the black hole and CFT entropies agree in these cases. The second line of (2), however, is much more context dependent and further can only be computed explicitly when we define Z to be a supersymmetric partition function (an index).

In Sect. 2, we begin with pure gravity in asymptotically AdS_3 spacetimes. We review how to properly define the gravitational action by including boundary terms, and then review the construction of the boundary stress tensor dual to the stress tensor of the CFT. In two-derivative gravity, this stress tensor obeys a Virasoro algebra with the central charge of Brown and Henneaux [9]. We then generalize to higher

[2] Here we mean that we can consider non-supersymmetric black hole solutions to an underlying supersymmetric theory. Corrections can be computed also for theories with no underlying supersymmetry (as we will discuss in Sect. 2) but explicit knowledge of the full Lagrangian is needed in these cases.

derivative theories of gravity and show how to obtain the generalized central charge. It turns out that the central charge can be found by a simple extremization principle. With these results in hand, we turn to computing the entropy of BTZ black holes in general higher derivative theories. A crucial role here is played by the construction of BTZ as a quotient of AdS, and its relation to a thermal AdS geometry via a modular transformation. This analysis will establish the agreement between the black hole and CFT entropies once the central charges have been shown to agree.

In Sect. 3 we add gauge fields into the mix and show how these are dual to currents in the boundary CFT. A central role is played by bulk Chern-Simons terms, since these turn out to completely determine the currents. Turning on flat connections for our gauge fields allows us to incorporate charged black holes. We then discuss the role of a Chern-Simons term for the gravitational field and show how it is used to deduce the difference between the central charges of the left and right moving sectors of the CFT.

The two specific string theory constructions that we will consider are reviewed in Sect. 4: the D1-D5 system giving rise to five-dimensional black holes with near horizon geometry $\mathrm{AdS}_3 \times S^3$, and wrapped M5-branes yielding four-dimensional black holes with near horizon geometry $\mathrm{AdS}_3 \times S^2$. To read off the exact central charges for these systems, we will use a combination of anomalies and supersymmetry. In particular, this will allow us to derive the exact corrections to the classical central charges, and hence derive a class of corrections to the black hole area law. The main emphasis is in showing how these exact results can be obtained even without knowing the explicit form of all higher derivative terms in the spacetime action. A nice application of this formalism is to small black holes dual to fundamental heterotic strings, and we will show how to derive the worldsheet central charges from gravity.

In Sects. 5, 6, 7, 8 we turn to the computation of the full partition function from the gravitational point of view. In order to have a chance of making an exact computation we focus on the elliptic genus, which is a particular partition function invariant under smooth deformations of the theory, due to bose-fermi cancellations. We review its main properties in Sect. 5, and then show in Sects. 6 and 7 how these properties emerge in the gravity description. Much of our discussion will follow the work of Dijkgraaf et al. [10] on the "Farey tail" description of the elliptic genus for the D1-D5 system. This gives a beautiful example of the matching between the CFT and gravity versions of the elliptic genus, including the effects of summing over geometries. Section 8 shows how to incorporate the effects of BPS excitations of top of the background geometries being summed over in the path integral. These include both supergravity fluctuations from Kaluza-Klein reduction, as well as non-perturbative brane states. In our brief discussion of the latter, following [11] we note how the appearance of both branes and anti-brane BPS states leads to the OSV formula [12] relating the AdS partition function to that of the topological string.

1.1 General References

In these lectures, we focus on one particular aspect of black hole physics, namely, the computation of the entropy/partition function of AdS_3 black holes, and our presentation is based mainly on [13, 14]. There are of course many other major issues that we will not touch on substantially, such as the information paradox, the black hole singularity, and black holes in other dimensions. There are a number of excellent pedagogical treatments of various aspects of black holes in string theory that complement the material discussed here. An incomplete list is [15, 16, 17, 18, 19, 20, 21].

2 Gravity in Asymptotically AdS_3 Spacetimes

2.1 Action and Stress Tensor

In this section, we consider pure gravity in three dimensions in the presence of a negative cosmological constant. This theory is described by an Einstein-Hilbert action supplemented by boundary terms

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R - \frac{2}{\ell^2}\right) + I_{\text{bndy}}. \tag{3}$$

The need for, and explicit form of, the boundary terms in the action will become clear as we proceed. We work in Euclidean signature and follow the curvature conventions of Misner, Thorne, and Wheeler.

One solution of the equations of motion is AdS_3,

$$ds^2 = \left(1 + r^2/\ell^2\right)dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2. \tag{4}$$

AdS_3 is homogeneous space of constant negative curvature. It has maximal symmetry, the isometry group being $SL(2,\,\mathbb{C}) \cong SL(2,\,\mathbb{R})_L \times SL(2,\,\mathbb{R})_R$ as will be reviewed later. The metric (4) is written in so-called global coordinates that cover the entire manifold. AdS_3 will play the role of the vacuum of our theory, in that it has the lowest mass of any solution. In fact, we will see that it is natural to assign it the negative mass $M = -\frac{\ell}{8G}$.

A more general one-parameter family of solutions is the non-rotating BTZ black hole [5, 6],

$$ds^2 = \frac{\left(r^2 - r_+^2\right)}{\ell^2}dt^2 + \frac{\ell^2}{\left(r^2 - r_+^2\right)}dr^2 + r^2 d\phi^2. \tag{5}$$

After rotating to Lorentzian signature, it is evident that this describes a black with event horizon at $r = r_+$, and Bekenstein-Hawking entropy

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G}. \tag{6}$$

Note that if we set $r_+^2 = -\ell^2$ we recover (4). The black hole solution (5) can be further generalized by adding charge and rotation; we will have much more to say about this.

By examining the large r behavior, it is apparent that the solution (5) asymptotically approaches AdS_3. We now want to state the precise conditions under which a metric can be said to be asymptotically AdS_3. This is a standard type of question in general relativity, and can be approached from different viewpoints. Our focus will be on demanding the existence of a well-defined action and variational principle. A motivation for this from the point of view of AdS/CFT is that the action takes on a well-defined meaning as giving, in a suitable semiclassical limit, the partition function of the CFT; indeed this is essentially the fundamental definition of the AdS/CFT correspondence. We would also like to include as large a class of metrics as possible. Furthermore, we have the freedom to adjust the boundary terms in (3) to make the action finite and stationary when the Einstein equations are satisfied.

To analyze this problem, it is convenient to work in coordinates where the metric takes the form (Gaussian normal coordinates)

$$ds^2 = d\eta^2 + g_{ij}dx^i dx^j. \tag{7}$$

Here g_{ij} is an arbitrary function of x^i $(i = 1,2)$ and the radial coordinate η. The allowed values of η are unbounded from above, although there may be a minimal value imposed by smoothness considerations. Now, it is apparent that the action written in (3) will diverge due to the large η integration, and so we regulate the integral by imposing a cutoff at some fixed value of η, which we eventually hope to take to infinity.

In terms of (3) the bulk term in the action (3) appears as, after an integration by parts,

$$I_{EH} = \frac{1}{16\pi G}\int d^2x d\eta\sqrt{g}\left(R^{(2)} + (\mathrm{Tr}K)^2 - \mathrm{Tr}K^2 - 2\Lambda\right) - \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^2x\sqrt{g}\,\mathrm{Tr}K, \tag{8}$$

where $R^{(2)}$ is the Ricci scalar associated with g_{ij}. K is the extrinsic curvature, defined as

$$K_{ij} = \frac{1}{2}\partial_\eta g_{ij}. \tag{9}$$

All indices are raised and lowered by g_{ij} and its inverse.

The variation of the boundary term contains a contribution $\delta\partial_\eta g_{ij}$. This term spoils a variational principle in which we hold fixed the induced metric on $\partial\mathscr{M}$ but not its normal derivative. This is rectified by adding to the action the Gibbons-Hawking term

$$I_{GH} = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^2x\sqrt{g}\,\mathrm{Tr}K. \tag{10}$$

We now consider the variation of the action with respect to g_{ij}. The variation will consist of two terms: a bulk piece that vanishes when the equations of motion are satisfied, and a boundary piece. Assuming that the equations of motion are satisfied, a simple computation gives

$$\delta(I_{EH} + I_{GH}) = -\frac{1}{16\pi G}\int_{\partial\mathscr{M}} d^dx\sqrt{g}\,\left(K^{ij} - \mathrm{Tr}Kg^{ij}\right)\delta g_{ij}. \tag{11}$$

The boundary stress tensor (which in the AdS/CFT correspondence is dual to the CFT stress tensor) is defined in terms of the variation as

$$\delta I = \frac{1}{2}\int_{\partial \mathscr{M}} d^2x\sqrt{g}\,T^{ij}\delta g_{ij}, \tag{12}$$

and so we have at this stage

$$T^{ij} = -\frac{1}{8\pi G}\left(K^{ij} - \mathrm{Tr}K g^{ij}\right), \tag{13}$$

which is a result derived by Brown and York [22]. Although we derived this result in the coordinate system (3), the result (13) is valid in any coordinate system, where g_{ij} is the induced metric on the boundary, and K_{ij} is the extrinsic curvature.

We now incorporate the specific features of asymptotically AdS spacetimes, which will require the addition of a second boundary term. From the basic solutions (4) and (5) it is evident that we should allow metrics that grow as r^2 at infinity. Translating to the η coordinate, this implies a growth $e^{2\eta/\ell}$. By studying the Einstein equations one finds that the general solution has subleading terms down by powers of $e^{-2\eta/\ell}$. We therefore write a "Fefferman-Graham expansion" [23] for the metric as

$$g_{ij} = e^{2\eta/\ell}g^{(0)}_{ij} + g^{(2)}_{ij} + \dots. \tag{14}$$

Omitted terms fall off at least as $e^{-\eta/\ell}$. $g^{(0)}_{ij}$ is the "conformal boundary metric"; it is clearly defined only up to Weyl transformations induced by a redefinition of η. It is this metric that we wish to identify with the metric of the boundary CFT.

Given $g^{(0)}_{ij}$, the subleading terms in the expansion (14) are found by solving Einstein's equations. Here we just note the following important relation that arises (see e.g. [24]):

$$\mathrm{Tr}(g^{(2)}) = \frac{1}{2}\ell^2 R^{(0)}, \tag{15}$$

where indices are lowered and raised with $g^{(0)}_{ij}$ and its inverse $g^{(0)ij}$.

Upon removal of the large η regulator, it is clear that the $g^{(0)}_{ij}$ plays the role of the boundary metric, and so it is natural to seek a variational principle in which $g^{(0)}_{ij}$ is held fixed, while the subleading parts of (14) are allowed to vary.[3] However, our action $I_{EH} + I_{GH}$ fails on two counts. First, using (11) it is not hard to check that the variation of $g^{(2)}_{ij}$ appears explicitly, and second that (11) diverges in the large η limit. Both these problems are solved by adding to the action the "counterterm" [25, 26]

$$I_{ct} = -\frac{1}{8\pi G\ell}\int_{\partial \mathscr{M}} d^2x\sqrt{g}. \tag{16}$$

[3] In higher dimensional AdS spacetimes a finite number of subleading terms are determined *algebraically* in terms of $g^{(0)}_{ij}$ and are therefore also kept fixed. See, e.g., [24]

Once this is included, it is straightforward to check that the on-shell variation of the action takes the form

$$\delta I = \frac{1}{2}\int d^2x\sqrt{g^{(0)}}\,T^{ij}\delta g^{(0)}_{ij}, \tag{17}$$

with

$$T_{ij} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{ij} - \mathrm{Tr}(g^{(2)})g^{(0)}_{ij}\right). \tag{18}$$

This is our AdS_3 stress tensor. The stress tensors for higher dimensional spacetimes can be found in the literature [26, 27], and additional related work appears in [28, 29, 30, 31, 32].

Note that the stress tensor has a nonzero trace [25],

$$\mathrm{Tr}(T) = -\frac{1}{8\pi G\ell}\mathrm{Tr}(g^{(2)}) = -\frac{\ell}{16\pi G}R^{(0)}, \tag{19}$$

where we used (15). This is the Weyl anomaly. In fact, the stress tensor defined here obeys all the properties of a stress tensor in CFT and we can thereby read off the central charge by comparing to the standard form of the Weyl anomaly, $\mathrm{Tr}(T) = -\frac{c}{24\pi}\,R$. This gives the central charge originally derived by Brown and Henneaux [9],

$$c = \frac{3\ell}{2G}. \tag{20}$$

In the absence of the Weyl anomaly, we can think of $g^{(0)}$ as specifying a conformal class of metrics, and the action is independent of the particular representative we choose. But when the Weyl anomaly is nonvanishing, we need to choose a specific representative. Another way to understand the Weyl anomaly is that although we succeeded in making the variation (17) finite, it is not hard to check that the action itself can suffer from a divergence linear in η. To cancel this divergence we are forced to add another counterterm that depends explicitly (and linearly) on our large η cutoff. The Weyl anoomaly can then be read off from the transformation of this term under a shift in the cutoff.

2.2 Virasoro Generators

To simplify the discussion, it is now convenient to take $g^{(0)}_{ij}$ to be a flat metric on the cylinder and to work in complex coordinates. We thus take $g^{(0)}_{ij}dx^i dx^j = dwd\overline{w}$ with $w \cong w+2\pi$. When we write $w = \sigma_1 + i\sigma_2$ we will think of σ_2 as the imaginary time direction. The stress tensor now has components

$$T_{ww} = \frac{1}{8\pi G\ell}g^{(2)}_{ww}, \quad T_{\overline{ww}} = \frac{1}{8\pi G\ell}g^{(2)}_{\overline{ww}}. \tag{21}$$

T_{ww} ($T_{\overline{ww}}$) is holomorphic (anti-holomorphic) as a consequence of the Einstein equations.

The Virasoro generators are defined in the usual fashion as contour integrals

$$\begin{aligned} L_n - \frac{c}{24}\delta_{n,0} &= \oint dw\, e^{-inw} T_{ww} \\ \tilde{L}_n - \frac{\tilde{c}}{24}\delta_{n,0} &= \oint d\overline{w}\, e^{in\overline{w}} T_{\overline{w}\overline{w}}. \end{aligned} \tag{22}$$

Looking ahead, we have allowed for an independent rightmoving central charge $\tilde{c}$, although at this stage $c = \tilde{c}$. The generators obey the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{23}$$

and likewise for the $\tilde{L}_n$. To establish this one studies the transformation of the stress tensor under the coordinate transformations that preserve the form of $g^{(0)}$. The infinitesimal transformation law is then used to derive the algebra (23).

Mass and angular momentum in AdS_3 are related to the Virasoro charges as

$$L_0 - \frac{c}{24} = \frac{1}{2}(M\ell - J), \quad \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{1}{2}(M\ell + J). \tag{24}$$

As a simple example consider the BTZ metric (5). We find $g^{(2)}_{ww} = g^{(2)}_{\overline{w}\overline{w}} = r_+^2/4$, and hence

$$L_0 = \tilde{L}_0 = \frac{\ell}{16G}\left(1 + \frac{r_+^2}{\ell^2}\right), \tag{25}$$

or

$$M = \frac{r_+^2}{8G\ell^2}, \quad J = 0. \tag{26}$$

Note that the pure AdS_3 metric (4) has $L_0 = \tilde{L}_0 = 0$, which is simply a consequence of its invariance under the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ group of isometries generated by $L_{0,\pm 1}$ and $\tilde{L}_{0,\pm 1}$.

2.3 Generalization to Higher Derivative Theories [13, 33, 34]

In the preceding we have been working with a two derivative action. In the context of string theory, or any other sensible approach to quantum gravity, this will just be the leading part of a more general effective action containing terms with arbitrary numbers of derivatives. If we are to make precise statements about such physical quantities as black hole entropy we need a systematic way of including the effect of higher derivative terms. For example, we no longer expect the entropy-area relation $S = A/4G$ to hold in the general case. On the face of it, even if we knew the explicit form of the action it would seem to be highly nontrivial to repeat the previous analysis and extract physical quantities. But in fact the problem is much easier than it first appears.

Using the fact that in three dimensions the Riemann tensor can be expressed in terms of the Ricci tensor, we may write an arbitrary higher derivative action as

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\,\mathscr{L}(g^{\mu\nu},\nabla_\mu,R_{\mu\nu}) + I_{bndy}. \tag{27}$$

In fact, there is one additional term that can be added, a gravitational Chern-Simons term related to the possibility of $c \neq \tilde{c}$ that is being suppressed in (27). We will come back to it later.

We now ask how to derive the generalized version of the central charge formula (20). Because AdS_3 is maximally symmetric we know that it will be a solution of our higher derivative theory, but we need to determine the length scale ℓ. To proceed, we write pure AdS_3 in the following coordinates

$$ds^2 = \ell^2(d\eta^2 + \sinh^2\eta\, d\Omega_2^2), \tag{28}$$

so that ℓ only appears as an overall factor. In (28) ℓ is of course a constant, but to determine its value it is useful to consider a *local* variation of compact support, $\ell \to \ell + \delta\ell(x)$. When the equations of motion are satisfied the action should be stationary under such a variation. The variation of the action computed around (28) takes a very simple form as follows from the fact that all tensorial quantities are covariantly constant on AdS_3. A moment's thought then shows that the variation takes the form

$$\delta I = \frac{1}{16\pi G}\int d^3x\,\frac{\partial}{\partial\ell}(\sqrt{g}\mathscr{L})\,\delta\ell(x). \tag{29}$$

So the equations of motion imply that $\sqrt{g}\mathscr{L}$ should be at an extremum with respect to *rigid* variations of ℓ. Given the explicit form of $\mathscr{L}$ we then need "only" solve an algebraic equation to determine ℓ.

Now we turn to the determination of the central charge. Conformal invariance implies the general relation $\mathrm{Tr}\,(T) = -\frac{c}{24\pi}R^{(0)}$. Consider (17) in the context of an infinitesimal Weyl transformation, $\delta g_{ij}^{(0)} = 2\delta\omega g_{ij}^{(0)}$, applied to a metric whose boundary is conformal to S^2,

$$\delta I = \frac{1}{2}\int d^2x\sqrt{g^{(0)}}\,T^{ij}\delta g_{ij}^{(0)} = -\frac{c}{24\pi}\delta\omega\int d^2x\sqrt{g^{(0)}}\,R^{(0)} = -\frac{c}{3}\delta\omega. \tag{30}$$

To extract c we evaluate (27) on the metric (28). $\mathscr{L}$ is a constant on this solution since AdS_3 is homogeneous, and so

$$I = \frac{\ell^3\mathscr{L}}{4G}\int d\eta\;\sinh^2\eta + I_{bndy}. \tag{31}$$

The integration is divergent at large η and so we impose a cutoff $\eta \leq \eta_{max}$ and write $\int d\eta\sinh^2\eta = -\frac{1}{2}\eta_{max} + \frac{1}{4}\sinh(2\eta_{max})$. Now, I_{bndy} is built out of the induced metric on the boundary. Assuming it is local, we can arrange it to subtract off the $\sinh(2\eta_{max})$ term but not the linear term. Indeed, as we discussed below (20) the linear divergence is the Weyl anomaly, which (like all anomalies) cannot be subtracted by local counterterms. So even after adding I_{bndy} the action diverges as

$$I_{div} = -\frac{\ell^3 \mathscr{L}}{8G}\eta_{max}. \tag{32}$$

Next, observe that a shift of η_{max} implements a Weyl transformation, $\delta\omega = \delta\eta_{max}$. We can therefore equate (30) with the variation of (32) to obtain[4] [13, 33]

$$c = \frac{3\ell^3 \mathscr{L}}{8G}. \tag{33}$$

Recall that $\mathscr{L}$ should be evaluated at the extremum of $\sqrt{g}\mathscr{L}$. But given (28) we see that $\sqrt{g}\mathscr{L} \propto \ell^3 \mathscr{L}$, so we can equally well say that we are extremizing c. We have now derived the *c-extremization principle*: the central charge is obtained by the value of (33) at its extremum.

Another version of (33) is also useful. Extremization of $\sqrt{g}\mathscr{L}$ implies

$$3\mathscr{L} + 2\ell^2 \frac{\partial \mathscr{L}}{\partial \ell^2} = 0. \tag{34}$$

Since all covariant derivatives vanish on the background we can ignore them for the purposes of this computation and write $\mathscr{L} = \mathscr{L}(g^{\mu\nu}, R_{\mu\nu})$. Given (28) we have that ℓ^2 appears in $g^{\mu\nu}$ but not in $R_{\mu\nu}$. This together with the fact that all indices in $\mathscr{L}$ must be contracted, implies

$$\ell^2 \frac{\partial \mathscr{L}}{\partial \ell^2} = -R_{\mu\nu}\frac{\partial \mathscr{L}}{\partial R_{\mu\nu}} = -\frac{2}{\ell^2} g_{\mu\nu}\frac{\partial \mathscr{L}}{\partial R_{\mu\nu}}, \tag{35}$$

where we also used $R_{\mu\nu} = \frac{2}{\ell^2} g_{\mu\nu}$ for (28). Using (34) and (35) we can rewrite (33) as [13, 34]

$$c = \frac{\ell}{2G} g_{\mu\nu}\frac{\partial \mathscr{L}}{\partial R_{\mu\nu}}. \tag{36}$$

This is the most convenient form for the AdS_3 central charge. As a quick check, if we return to the action (3) we find $\frac{\partial \mathscr{L}}{\partial R_{\mu\nu}} = g^{\mu\nu}$ and so we recover the Brown-Henneaux central charge, $c = 3\ell/2G$.

We will now show how to use this result to derive the entropy of a BTZ black hole in a general higher derivative theory of gravity.

2.4 Thermal AdS Partition Function

The AdS/CFT correspondence is fundamentally a relation between partition functions

$$Z_{AdS}(g^{(0)}) = Z_{CFT}(g^{(0)}). \tag{37}$$

[4] Note that Ref. [33] also considers the effect of higher derivatives on conformal anomalies in higher dimensions.

Here, we have just indicated the dependence on the metric, although more generally other data will enter in as well. Modulo the Weyl anomaly, $g^{(0)}$ labels a conformal class of boundary metrics.

In this section, we will consider the case in which $g^{(0)}$ is the flat metric on a torus of modular parameter τ. We write the line element of the boundary in complex coordinates as $ds^2 = dwd\overline{w}$, with

$$w \cong w + 2\pi \cong w + 2\pi\tau. \tag{38}$$

Z_{CFT} can either be evaluated as a path integral on the torus (assuming that there exists a Lagrangian formulation of the theory) or in the canonical formulation as

$$Z_{CFT}(\tau,\overline{\tau}) = \mathrm{Tr}\left[e^{2\pi i\tau(L_0-\frac{c}{24})}e^{-2\pi i\overline{\tau}(\tilde{L}_0-\frac{\tilde{c}}{24})}\right]. \tag{39}$$

If fermions are present in the theory, we also need to specify their periodicities around the two cycles of the torus. As written, (39) implies anti-periodic boundary conditions around the time circle; periodic fermions are incorporated by including $(-1)^F$ in the trace, where F is the fermion number. By trading the Virasoro charges for mass and angular momentum according to (24), we see that the imaginary part of τ plays the role of inverse temperature, while the real part is a chemical potential for angular momentum.

Now consider Z_{AdS}. We can again write a canonical formula like (39), but now its implementation is problematic since we lack a satisfactory description of the Hilbert space in the gravitational language. At low energies, the Hilbert space is well understood as comprising a gas of particles moving on AdS, but at sufficiently high energies we encounter black hole solutions. Black hole solutions are clearly not to be interpreted as individual states of the theory (since they carry entropy), and so it is not altogether clear how to include them in the trace. The situation is more satisfactory in the path integral formulation, where we can include the black holes as additional saddle points of the functional integral, weighted by their action. Ultimately, since Z_{CFT} *is* well defined, we hope to use it to shed light on the Hilbert space of the gravitational theory, including the black hole regime.

We will therefore attempt to make sense of

$$Z_{AdS}(\tau,\overline{\tau}) = \sum e^{-I}. \tag{40}$$

The summation is supposed to run over all saddle points of the full effective action I (which in principle includes all corrections coming from string and loop corrections), such that the boundary metric has modular parameter τ. One subtle point, whose importance will become clear as we proceed, is that certain saddle points that are just coordinate transformations of other saddle points will appear as distinct terms in the summation. This is the analogue of the fact that in ordinary gauge theories we can treat gauge transformations that are nontrivial on the boundary as global symmetries: we are not forced to demand that physical states are invariant under such gauge transformations.

Another important point is that we will regard I as capturing the complete *local* part of the effective action. This action can in principle be computed in flat spacetime and then evaluated on the asymptotically AdS backgrounds appearing in (40). But this is not the whole story, as can be appreciated intuitively by thinking in terms of field theory Feynman diagrams. Geometries contributing to (40) have a periodic imaginary time direction, and there are contributions from Feynman diagrams that wind around the time direction. Such diagrams clearly are not incorporated in the local effective action I. Instead, we have to incorporate their effects as additional saddle points in (40). In fact, there is a clean way of isolating these effects via their behavior in τ. Local terms contribute to $\ln Z$ linearly in τ and $\overline{\tau}$, while the nonlocal terms are exponentially suppressed for $\mathrm{Im}(\tau) \to 0$. We will see this explicitly as we proceed.

The simplest saddle point is just pure AdS_3 suitably identified. We take the AdS_3 metric (4) and define $w = \phi + it/\ell$, with w identified as in (38). We know on account of maximal symmetry that this is a saddle point of I, even taking into account all higher derivative corrections. What is the value of I evaluated on this solution? Since we do not know the explicit form of I we need to proceed indirectly. The idea is to integrate (17). To use (17) we need to work in coordinates with fixed periodicity, so we define

$$z = \frac{i - \overline{\tau}}{\tau - \overline{\tau}} w - \frac{i - \tau}{\tau - \overline{\tau}} \overline{w}, \tag{41}$$

obeying $z \cong z + 2\pi \cong z + 2\pi i$. τ now appears in the metric,

$$ds^2 = \left| \frac{1 - i\tau}{2} dz + \frac{1 + i\tau}{2} d\overline{z} \right|^2 . \tag{42}$$

Writing out (17) in the z coordinates, and then converting back to w coordinates gives

$$\delta I = 4\pi^2 i \left(-T_{ww} \delta\tau + T_{\overline{w}\overline{w}} \delta\overline{\tau} \right). \tag{43}$$

Using that $L_0 = \tilde{L}_0 = 0$ for this geometry, we know from (22) that

$$T_{ww} = -\frac{c}{48\pi}, \quad T_{\overline{w}\overline{w}} = -\frac{\tilde{c}}{48\pi}. \tag{44}$$

This yields the action

$$I_{thermal} = \frac{i\pi}{12} (c\tau - \tilde{c}\overline{\tau}). \tag{45}$$

We can summarize the above computation as saying that we have determined the exact low temperature behavior of Z_{AdS}. In particular, as $\mathrm{Im}(\tau) \to \infty$ we have

$$\ln Z_{AdS}(\tau, \overline{\tau}) = -\frac{i\pi}{12} (c\tau - \tilde{c}\overline{\tau}) + (\text{exponentially suppressed terms}). \tag{46}$$

This conclusion follows since we have incorporated all local terms in the effective action, along with the fact that L_0 and $\tilde{L}_0$ have a gap in their spectrum above 0 (this in turn follows from the fact that AdS effectively acts like a finite size box.)

The exponentially suppressed terms are down by at least $e^{2\pi i(\Delta\tau-\tilde{\Delta}\bar{\tau})}$, where Δ is the gap in the L_0 spectrum. The contribution of these suppressed terms depends on the precise theory under consideration (e.g. on the field content in addition to gravity), and so we postpone incorporating them until later.

We'll now show that the high temperature behavior of the partition function is governed by black holes. To illustrate the basic point in the simplest context, let us consider the non-rotating black hole metric (5). In order to avoid a conical singularity at $r = r_+$ we need to make the identification $t \cong t + 2\pi\ell^2/r_+$. In other words, $\tau = i\ell/r_+$. Note that we have thereby identified the Hawking temperature as $T = r_+/(2\pi\ell^2)$. So for τ purely imaginary the non-rotating black hole metric contributes to the partition function as long as we set $r_+ = i\ell/\tau$.

Next, we need to compute the action of the black hole to see if and when it dominates the thermal AdS geometry. As we will now show, after a judicious change of coordinates the needed computation becomes equivalent to that yielding (45). We define

$$w' = -\frac{w}{\tau}, \quad r' = \frac{\ell}{r_+}\sqrt{r^2 - r_+^2} \tag{47}$$

with $w' = \phi' + it'/\ell$. Then, the black hole metric (45) becomes

$$ds^2 = (1 + r'^2/\ell^2)dt'^2 + \frac{\ell^2 dr'^2}{1 + r'^2/\ell^2} + r'^2 d\phi'^2, \tag{48}$$

which is just the pure AdS_3 metric. But now we have the identifications $w' \cong w' + 2\pi \cong w' + 2\pi\tau'$, with $\tau' = -1/\tau$. In other words, we have shown the equivalence (up to coordinate transformation) of thermal AdS with modular parameter τ and a black hole with modular parameter $\tau' = -1/\tau$:

$$\text{Thermal AdS with } \tau \quad \Leftrightarrow \quad \text{Black hole with } \tau' = -1/\tau. \tag{49}$$

Of course, so far we have only established this for pure imaginary τ, but we will generalize in due course.

Now, the action is invariant under the coordinate transformation (47), so we can immediately conclude that the black hole action is

$$I_{BTZ} = \frac{i\pi}{12}(c\tau' - \tilde{c}\bar{\tau}') = -\frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{\tilde{c}}{\bar{\tau}}\right). \tag{50}$$

This results shows that at high temperature, $\mathrm{Im}(\tau) \to 0^+$, the black hole has less action than thermal AdS and hence will dominate the partition function.

Equation (50) gives the exact high temperature behavior of the partition function; specifically, the part of $\ln Z$ linear in τ^{-1},

$$\ln Z = \frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{\tilde{c}}{\bar{\tau}}\right) + (\text{exponentially suppressed terms}). \tag{51}$$

Let us use this derive an expression for the entropy S at high temperature. From (39) we can write in the saddle point approximation

$$\ln Z = S + 2\pi i\tau\left(L_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right). \tag{52}$$

We further have

$$\begin{aligned} L_0 - \frac{c}{24} &= \frac{1}{2\pi i}\frac{\partial \ln Z}{\partial \tau} = -\frac{c}{24\tau^2} \\ \tilde{L}_0 - \frac{\tilde{c}}{24} &= -\frac{1}{2\pi i}\frac{\partial \ln Z}{\partial \bar{\tau}} = -\frac{\tilde{c}}{24\bar{\tau}^2}. \end{aligned} \tag{53}$$

From this we read off the entropy as

$$S = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24})} + 2\pi\sqrt{\frac{\tilde{c}}{6}\left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right)}. \tag{54}$$

This is the Cardy formula. This formula in fact gives the high temperature behavior of the entropy of any CFT, assuming unitarity and a gapped spectrum of L_0 starting at 0. The standard derivation of the Cardy formula is based on modular invariance and precisely parallels the gravitational approach adopted here; so the agreement between the gravity and CFT sides is unsurprising. Indeed, this shows that the high temperature entropy is guaranteed to agree between the two sides provided that the central charges agree.

In the context of two-derivative gravity, formula (54) can be written as $S = A/4G$. But with higher derivative terms included this is no longer the case. More generally, we have Wald's entropy formula [35, 36, 37, 38]

$$S = -\frac{1}{8G}\int_{hor} dx\sqrt{h}\frac{\delta\mathscr{L}}{\delta R_{\mu\nu\alpha\beta}}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}. \tag{55}$$

In fact, (54) and (55) are equivalent.[5] This can be shown using (36); for details, the reader is directed to [13, 34].

Consider the restricted case in which $c = \tilde{c}$. Recall that the value of c is determined by extremizing the function (33). We can translate this into an extremization principle for the entropy. Specifically, S is determined by extremizing the function (54) while holding fixed $L_0 - \frac{c}{24}$ and $\tilde{L}_0 - \frac{\tilde{c}}{24}$ (from (24) this is the same as holding fixed the dimensionless mass and angular momentum).

Next, to set the stage for a more general discussion let us examine the relation between thermal AdS$_3$ and the BTZ black hole from a more geometrical perspective. Thermal AdS$_3$ clearly has the topology of a solid torus. The boundary is a two-dimensional torus. On the boundary torus there are two independent noncontractible cycles, which we can take to be $\Delta\phi = 2\pi$, and $\Delta t/\ell = -2\pi i\tau$ (we are just considering the case of pure imaginary τ for the moment). Now consider allowing these cycles to move off the boundary torus into the bulk geometry. It is then clear that the ϕ cycle is contractible in the bulk while the t cycle is not.

[5] Actually, (54) is a bit more general in that (55) only applies when $c = \tilde{c}$.

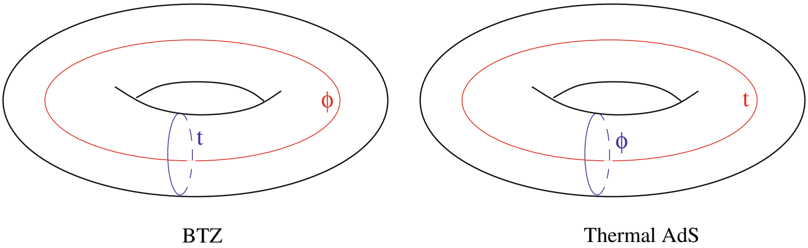

Fig. 1 Relation between BTZ and thermal AdS_3

The coordinate transformation (47) that relates thermal AdS_3 to BTZ interchanges ϕ and t, so for BTZ we find that it is the t cycle that is contractible in the bulk, while the ϕ cycle is noncontractible. This is illustrated in Fig. 1.

The generalization of this story involves rotating black holes. To deal with this efficiently it is advantageous to describe the BTZ black hole as a quotient of AdS.

2.5 BTZ Black Holes as Quotients [39]

Euclidean AdS_3 can be written

$$ds^2 = \frac{d\rho^2 + dz d\bar{z}}{\rho^2}. \tag{56}$$

Now consider the matrix

$$g = \begin{pmatrix} \rho + z\bar{z}/\rho & z/\rho \\ \bar{z}/\rho & 1/\rho \end{pmatrix}. \tag{57}$$

$\det g = 1$, so $g \in SL(2,\,\mathbb{C})$. Actually, g can be written as $g = hh^\dagger$ with $h \in SL(2,\,\mathbb{C})$. Since g is invariant under $h \to hf$ with $f \in SU(2)$, we see that the space of g matrices can be identified with the coset $SL(2,\,\mathbb{C})/SU(2)$. The line element (56) is the same as the natural line element on the coset,

$$ds^2 = \frac{1}{2}\mathrm{Tr}(g^{-1}dgg^{-1}dg). \tag{58}$$

Since this is invariant under $h \to \alpha h,\ \alpha \in SL(2,\,\mathbb{C})$, we see that Euclidean AdS_3 has an $SL(2,\,\mathbb{C})$ group of isometries.

The BTZ black hole is obtained from AdS_3 by making $SL(2,\,\mathbb{C})$ identifications, $h \cong \gamma h$. Since we can always redefined h as $h = \alpha h'$, we see that γ is only defined up to conjugation by $SL(2,\,\mathbb{C})$. So without loss of generality we can take γ to be diagonal and write

$$\gamma = \begin{pmatrix} e^{-i\pi\tau} & 0 \\ 0 & e^{i\pi\tau} \end{pmatrix}. \tag{59}$$

In terms of coordinates, this implies the identification $(\rho, z) \cong (e^{-i\pi(\tau-\bar{\tau})}\rho,\ e^{-2\pi i\tau}z)$. Now write $z = e^{-iw}$ so that at the boundary $(\rho = 0)$ we have the identification $w \cong w + 2\pi\tau$. This identifies τ in (59) as the modular parameter of the boundary torus. In other words, quotienting AdS_3 by γ yields thermal AdS_3:

$$ds^2 = \left(1 + r^2/\ell^2\right) dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2 \tag{60}$$

with the identification $w \cong w + 2\pi \cong w + 2\pi\tau$, where $w = \phi + it/\ell$.

To construct an "$SL(2,\ \mathbb{Z})$ family of black holes" [10, 40], we consider the modular transformed version of (59)

$$\gamma = \begin{pmatrix} e^{-i\pi\frac{a\tau+b}{c\tau+d}} & 0 \\ 0 & e^{i\pi\frac{a\tau+b}{c\tau+d}} \end{pmatrix}, \tag{61}$$

with $(a,b,c,d) \in \mathbb{Z}$ and $ad - bc = 1$. This is a geometry whose conformal boundary has modular parameter $\frac{a\tau+b}{c\tau+d}$. But, as we illustrated in a simplified context above, if we change coordinates we can bring the modular parameter back to τ. The action for this geometry can be read off from (45),

$$I(\tau, \bar{\tau}) = \frac{i\pi}{12}\left[c\left(\frac{a\tau+b}{c\tau+d}\right) - \tilde{c}\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)\right]. \tag{62}$$

Note the unfortunate notation in which the same symbol c appears with two different meanings. These $SL(2,\ \mathbb{Z})$ black holes will make an appearance later as saddle points of the Euclidean path integral.

For completeness, we will explicitly write the metric of the rotating BTZ black hole, generalizing (5). Start from (56) and write

$$\begin{aligned} z &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+ + r_-}{\ell}(\phi + it/\ell)\right\} \\ \rho &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+}{\ell}\phi + ir_- t/\ell\right\} \end{aligned} \tag{63}$$

with r_+ real and r_- imaginary. The metric is then

$$ds^2 = \frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{r^2\ell^2} dt^2 + \frac{\ell^2 r^2}{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)} dr^2 + r^2\left(d\phi + i\frac{r_+ r_-}{\ell r^2} dt\right)^2. \tag{64}$$

The modular parameter is

$$\tau = \frac{i\ell}{r_+ + r_-}. \tag{65}$$

The action is given by (50). The event horizon is at $r = r_+$ and the entropy is given by (54) with

$$L_0 - \frac{c}{24} = \frac{(r_+ - r_-)^2}{16G\ell}, \quad \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{(r_+ + r_-)^2}{16G\ell}. \tag{66}$$

To get the Lorentzian black hole we replace t by it and continue r_- to real values. Note that the Virasoro charges (64) are then real. Extremal black holes correspond to $r_- = \pm r_+$.

3 Charged Black Holes and Chern-Simons Terms

An important generalization is to allow our black holes to carry charge. We will see that this comes about in an elegant fashion. To begin, we will consider a collection of $U(1)$ gauge fields. Besides the usual Maxwell term, since we are working in an odd-dimensional spacetime our gauge fields can have Chern-Simons terms. We will find that the Chern-Simons terms are needed in order for the charge to be nonzero. Indeed, the charge comes *entirely* from the Chern-Simons terms. Later, we will indicate where these Chern-Simons originate from the higher dimensional string/M-theory viewpoint. Our presentation follows [14]. Additional relevant work on Chern-Simons theory includes [41, 42].

3.1 U(1) Gauge Fields in AdS$_3$

Let us first consider the case of a single $U(1)$ gauge field. A Chern-Simons term in three spacetime dimensions is, in differential form language,

$$I_{CS} = \frac{ik}{8\pi} \int_M AdA. \tag{67}$$

The normalization was chosen so that the constant k will be identified with the level of a corresponding current algebra. A gauge transformation is $\delta A = d\Lambda$. I_{CS} is not gauge invariant but instead varies by a boundary term:[6] $\delta I_{CS} = \frac{ik}{4\pi} \int_{\partial M} \Lambda dA$.

The gauge field admits an expansion analogous to (14),

$$A = A^{(0)} + e^{-2\eta/\ell} A^{(2)} + \dots, \tag{68}$$

and we choose the gauge $A_\eta = 0$. Analysis of the field equations (including the effect of Maxwell type terms) shows that $A^{(0)}$ is a flat connection; that is, the field strength corresponding to (3.2) falls off as $e^{-2\eta/\ell}$. For this reason, all the results we derive below for the currents and stress tensor will be valid in the presence of arbitrary higher derivative terms for the gauge fields. The flatness of $A^{(0)}$ implies that only the Chern-Simons terms yields nonzero boundary variations.

In analogy with (17) a boundary current is obtained from the on-shell variation of the action with respect to $A^{(0)}$,

[6] Note that the equations of motion will still be gauge invariant.

$$\delta I = \frac{i}{2\pi}\int_{\partial AdS} d^2x \sqrt{g^{(0)}} J^\alpha \delta A_\alpha^{(0)}. \tag{69}$$

We now need to define the appropriate variational principle that yields the equations of motion for our gauge field. For definiteness, consider the pure AdS geometry (4). Naively, one might guess that in the variational principle, one could hold fixed both $A_t^{(0)}$ and $A_\phi^{(0)}$. But this is too strong, since there will then typically be no smooth solutions of the equations of motion with the assumed boundary conditions. The issue is the holonomy around the contractible ϕ circle, expressed by $\int d\phi A_\phi$. When we contract the circle we need the holonomy to either vanish or match onto an appropriate source to avoid a singularity. So it is only $A_t^{(0)}$ that can take generic values. If we define $w = \phi + it/\ell$ as usual, then an appropriate variational principle is to hold fixed *either* $A_w^{(0)}$ or $A_{\overline{w}}^{(0)}$ but not both. The sign of k will determine which component to hold fixed.

Let us assume that k is positive. Then we claim that to derive the equations of motion we should demand that the action be stationary under variations that hold fixed $A_{\overline{w}}^{(0)}$. That is to say, we demand that (69) take the form

$$\delta I = \frac{i}{2\pi}\int_{\partial AdS} d^2w \sqrt{g^{(0)}} J^{\overline{w}} \delta A_{\overline{w}}^{(0)}. \tag{70}$$

But we can readily check that the variation of (67) does not take this form. However, we still have the freedom to add boundary terms to the action. If we add to the action the term

$$I_{gauge}^{bndy} = -\frac{k}{16\pi}\int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta} A_\alpha A_\beta \tag{71}$$

then the variation of the action *does* take the form (70) with

$$J_w = \frac{1}{2} J^{\overline{w}} = \frac{ik}{2} A_w^{(0)}. \tag{72}$$

Another way to say this is that $J_{\overline{w}} = 0$, which means that our current is purely left-moving.

Note that the boundary term (71) depends on the metric, and so will contribute to the stress tensor. This in contrast to the topological term (67). A straightforward computation yields

$$T_{\alpha\beta}^{gauge} = \frac{k}{8\pi}\left(A_\alpha^{(0)} A_\beta^{(0)} - \frac{1}{2} A^{(0)\gamma} A_\gamma^{(0)} g_{\alpha\beta}^{(0)}\right), \tag{73}$$

or, in complex coordinates,

$$\begin{aligned} T_{ww}^{gauge} &= \frac{k}{8\pi} A_w^{(0)} A_w^{(0)}, \\ T_{\overline{w}\overline{w}}^{gauge} &= \frac{k}{8\pi} A_{\overline{w}}^{(0)} A_{\overline{w}}^{(0)}, \\ T_{w\overline{w}}^{gauge} &= T_{\overline{w}w}^{gauge} = 0. \end{aligned} \tag{74}$$

The index (0) on the gauge field reminds us that boundary expressions strictly refer to just the leading term in the expansion (68) for the bulk gauge field. In the following, we will reduce clutter by dropping this index.

We can now see why the sign of k is important; if we took k negative (74) would imply that the energy is unbounded below. The case of k negative needs to handled differently, by flipping the sign of the boundary term (71). The same analysis then yields a purely rightmoving current.

Turning to the general case of multiple $U(1)$ gauge fields, we write the action as

$$I = \frac{i}{8\pi}\int d^3x\left(k^{IJ}A_I dA_J - \tilde{k}^{IJ}\tilde{A}_I d\tilde{A}_J\right) - \frac{1}{16\pi}\int_{\partial AdS} d^2x\sqrt{g}g^{\alpha\beta}\left(k^{IJ}A_{I\alpha}A_{J\beta} + \tilde{k}^{IJ}\tilde{A}_{I\alpha}\tilde{A}_{J\beta}\right). \quad (75)$$

Both k^{IJ} and $\tilde{k}^{IJ}$ are symmetric matrices with positive eigenvalues. The IJ indices on k^{IJ} versus $\tilde{k}^{IJ}$ are independent, and so can take different ranges.

In conformal gauge, the gauge fields contribute to the currents and stress tensor as,

$$\begin{aligned}
T^{gauge}_{ww} &= \frac{1}{8\pi}k^{IJ}A_{Iw}A_{Jw} + \frac{1}{8\pi}\tilde{k}^{IJ}\tilde{A}_{Iw}\tilde{A}_{Jw}, \\
T^{gauge}_{\overline{w}\overline{w}} &= \frac{1}{8\pi}k^{IJ}A_{I\overline{w}}A_{J\overline{w}} + \frac{1}{8\pi}\tilde{k}^{IJ}\tilde{A}_{I\overline{w}}\tilde{A}_{J\overline{w}}, \\
T^{gauge}_{w\overline{w}} &= T^{gauge}_{\overline{w}w} = 0, \\
J^I_w &= \frac{i}{2}k^{IJ}A_{Jw}, \quad J^I_{\overline{w}} = 0, \\
\tilde{J}^I_w &= 0, \quad \tilde{J}^I_{\overline{w}} = \frac{i}{2}\tilde{k}^{IJ}\tilde{A}_{J\overline{w}}.
\end{aligned} \quad (76)$$

The modes of the currents are defined as

$$J^I_n = \oint\frac{dw}{2\pi i}e^{-inw}J^I_w, \quad \tilde{J}^I_n = -\oint\frac{d\overline{w}}{2\pi i}e^{in\overline{w}}\tilde{J}^I_{\overline{w}}. \quad (77)$$

By writing out the formulas for the changes in the stress tensor and currents under a variation of the gauge field, we can infer the commutation relations

$$\begin{aligned}
\left[L_m, J^I_n\right] &= -nJ^I_{m+n} \\
\left[J^I_m, J^J_n\right] &= \frac{1}{2}mk^{IJ}\delta_{m+n},
\end{aligned} \quad (78)$$

and likewise for the tilded generators.

3.2 Spectral Flow

Together with the Virasoro algebra (23), the algebra (78) admits a so-called spectral flow automorphism that will play an important role. For arbitrary parameters η_I the algebra is preserved under

$$L_n \to L_n + 2\eta_I J_n^I + k^{IJ}\eta_I\eta_J\delta_{n,0} \quad (79)$$
$$J_n^I \to J_n^I + k^{IJ}\eta_J\delta_{n,0}.$$

From our explicit formulas for the generators we see that this is equivalent to

$$A_{Iw} \to A_{Iw} + 2\eta_I. \quad (80)$$

This constant shift of the gauge potentials is equivalent to shifting the periodicities of charged fields. In particular, since the phase factor acquired by a particle of charge J_0^I taken around the AdS cylinder is

$$e^{iJ_o^I \oint dw A_{Iw}}, \quad (81)$$

we see that the shift (80) induces the the phase $e^{4\pi i J_i^I \eta_I}$.

From now on we will use another normalization for the gauge charges by defining

$$q^I = 2J_0^I, \quad \tilde{q}^I = 2\tilde{J}_0^I, \quad (82)$$

where the 2 is introduced for convenience.

3.3 Nonabelian Gauge Fields

Besides the $U(1)$ gauge fields, in the main cases of interest from the string theory perspective we will also have $SU(2)$ gauge fields. From the higher dimensional point of view we will be considering either $AdS_3 \times S^2$ or $AdS_3 \times S^3$ geometries. The spheres have isometry group $SO(3) \cong SU(2)_R$ or $SO(4) \cong SU(2)_L \times SU(2)_R$, and we then have the associated Kaluza-Klein gauge fields. To show that these gauge fields have three-dimensional Chern-Simons terms is somewhat subtle but can be derived by a careful consideration of the background flux configuration that supports the sphere [43, 44, 45].

The $SU(2)_L$ Chern-Simons term is

$$I_{CS} = -\frac{ik}{4\pi}\int d^3x \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right), \quad (83)$$

with $A = A^a \frac{i\sigma^a}{2}$. Invariance of the path integral under large gauge transformation fixes k to be an integer. The $SU(2)_R$ Chern-Simons term is taken with the opposite sign, as above. As before, in order to get purely left or right moving currents we need to add a boundary tern. This has the same form as (71) except that we sum over the group indices. We will just be considering solutions in which $A^{(0)a}$ and $\tilde{A}^{(0)a}$ are nonvanishing only for $a = 3$. We can then easily incorporate the corresponding currents into the previous discussion by extending the I index to include $I = 0$, and write

$$A^{(0)3} = A_{I=0}, \quad k = k^{00}, \quad k^{0,I>0} = k^{I>0,0} = 0, \quad (84)$$

and likewise for the tilded counterparts. All the formulas (76) now carry over.

3.4 Supersymmetry

We now discuss one important implication of supersymmetry. In two dimensions, we characterize the amount of supersymmetry by the number of left and right moving supercharges, $(N_L,\ N_R)$. Here the focus will be on theories with either (0,4) or (4,4) supersymmetry. Supersymmetry then implies the existence of an $SU(2)$ R-symmetry that rotates the supercharges into one another (the 4 supercharges transform as two doublets). In the (4,4) case we have $SU(2)_L \times SU(2)_R$ R-symmetry. The R-symmetry currents correspond to the $SU(2)$ gauge fields in (3.17).

Of central importance to us is that the supersymmetry algebra relates the level of the $SU(2)$ current algebra k to the central charge c as $c = 6k$. When we recall that k appears in (83), we see that determining the exact central charge is equivalent to determining the $SU(2)$ Chern-Simons term. Of course, in the (0,4) case this argument only gives us the right moving central charge, but we will see momentarily how a related argument gives the left moving central charge.

3.5 Gravitational Chern-Simons term [46]

The left and right moving central charges of a two-dimensional CFT are independent and need not be equal. However, if $c \neq \tilde{c}$ it is not possible to couple such a theory to gravity in a diffeomorphism invariant fashion: there is a gravitational anomaly [47, 48, 49]. To write the anomaly we can work in terms of the connection 1-forms, defined as $\Gamma^i{}_j = \Gamma^i_{jk} dx^k$, where Γ^i_{jk} are the usual Christoffel symbols. The breakdown of diffeomorphism invariance is signaled by the non-conservation of the stress tensor:[7]

$$\nabla_i T^{ij} = -i\frac{c-\tilde{c}}{96\pi}\, g^{ij}\epsilon^{kl}\partial_k\partial_m\Gamma^m_{il}. \tag{85}$$

Equivalently, under an infinitesimal diffeomorphism, $x^i \to x'^i = x^i - \xi^i(x)$, the effective is not invariant but instead changes by

$$\delta I_{eff} = -i\frac{c-\tilde{c}}{96\pi} \int \mathrm{Tr}(v d\Gamma), \tag{86}$$

with $v^i_j = \partial_j \xi^i$.

To reproduce this from the AdS point of view, we need a term in the bulk that varies under diffeomorphisms. Further, the variation should be a pure boundary term, otherwise the bulk theory will be inconsistent. Up to boundary terms, there is a unique possibility, the gravitational Chern-Simons term [50, 51, 52]:

$$I_{CS}(\Gamma) = -i\beta \int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right). \tag{87}$$

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[7] Alternatively, one can add a local counterterm to render the stress conserved but non-symmetric.

Under an infinitesimal diffeomorphism $\delta\Gamma = dv + [\Gamma,\ v]$, so

$$\delta I_{CS} = -i\beta \int_{\partial AdS} \mathrm{Tr}(v d\Gamma). \tag{88}$$

Comparing with (86) we read off

$$\beta = \frac{c-\tilde{c}}{96\pi}. \tag{89}$$

The full stress tensor in the presence of the gravitational Chern-Simons term is discussed in [46, 53, 54].

We now know how to compute both central charges of the (0,4) theory. The coefficient of the $SU(2)$ Chern-Simons term gives us $\tilde{c}$, while the gravitational Chern-Simons term gives us $c-\tilde{c}$. The reason why this is useful is that (at least in the cases we will consider) the Chern-Simons terms arise at tree level and one-loop, but receive no other corrections, and hence we can determine them exactly. In particular, we can determine the central charges without knowledge of the full effective action including higher derivative terms. This is to be contrasted with the non-supersymmetric case, where the c-extremization procedure discussed in Sect. 2.3, while efficient, does require the explicit action as an input.

4 String Theory Constructions

We will now review two standard constructions of AdS_3 geometries in string theory. The first, and best known, example is realized as the near horizon geometry of the D1-D5 system [1] (for reviews see [16, 19]). This yields $AdS_3 \times S^3 \times M_4$, where M_4 is T^4 or $K3$. The second example [55] is realized in terms of wrapped M5-branes, and yields $AdS_3 \times S^2 \times M_6$, with M_6 being $T^6,\ K3\times T^2$, or CY_3.

4.1 D1-D5 System

To describe the brane construction, we first work at weak string coupling and consider IIB string theory on $R^{4,1}\times S^1 \times M_4$. We wrap N_5 D5-branes on $S^1\times M_4$, and N_1 D1-brane on S^1. This setup preserves 8 of the original 32 supercharges. When the length scale associated with M_4 is small compared to the S^1 the low energy dynamics of the system is described by a theory on the $1+1$ dimensional intersection. The standard weak coupling open string quantization yields a $U(N_1)\times U(N_5)$ supersymmetric gauge theory, which flows to a nontrivial CFT in the infrared, with (4,4) susy. We want to know the left and right moving central charges. The easiest way to compute these is by using anomalies (see Sect. 5.3.1 of [16]). Focus on the left moving side, say. As we have discussed previously, the existence of 4 supercharges means that the CFT has an $SU(2)$ R-symmetry. The level k of the corresponding current algebra is related to the central charge by $c = 6k$. The current algebra also implies that the R-symmetry currents are anomalous when coupled to external gauge fields,[8]

$$D_{\overline{w}} J^a_w = \frac{ik}{2} \partial_w A^a_{\overline{w}}. \tag{90}$$

Chiral anomalies are related to topology and are invariant under smooth deformations of the theory. In our context, the level k is an integer, which can equally well be evaluated in the weak coupling gauge theory description valid in the UV. The anomaly arises from one loop diagrams. This is a fairly straightforward computation, and the result is that $k = N_1 N_5$. The same analysis holds for the right movers. We conclude that the exact central charges are $c = \tilde{c} = 6N_1N_5$. Note how little went into this result: We just needed to know that in the IR we have (4,4) susy, and that the IR theory is reached via RG flow from the UV gauge theory.

Next, we recall the anomaly inflow mechanism [56], which will be useful in relating the CFT central charges to those of the AdS_3 theory. Note that the D1-D5 system is localized at a point in the 4 noncompact spatial dimensions, and hence is invariant under the corresponding $SO(4)$ group of rotations. If we write $SO(4) \cong SU(2)_L \times SU(2)_R$, we identify the left and right moving $SU(2)$ R-symmetry groups. We can think of the $SO(4)$ as acting on the vector space normal to the brane worldvolume (the so-called normal bundle). We can further allow the $SO(4)$ rotations to vary over the worldvolume, which leads to an $SO(4)$ gauge theory. Now, we know from (90) that in general the worldvolume theory on the brane is not invariant under such local $SO(4)$ transformations, the effective action instead varies as

$$\delta I_{brane} = -\frac{i}{4\pi} \int d^2w \left(D_{\overline{w}} J^a_w \Lambda^a + D_w \tilde{J}^a_{\overline{w}} \tilde{\Lambda}^a \right), \tag{91}$$

where we have written the result in terms of the $SU(2)_L \times SU(2)_R$ parameters.

On the other hand, from the point of view of the full ten dimensional string theory these gauge transformations are just coordinate transformations (or, more accurately, local Lorentz transformations), but it is well known that the full theory is nonanomalous. Indeed, otherwise the IIB string theory would be inconsistent, since we are talking about a potential breakdown of diffeomorphism invariance. So something must be canceling the variation of the brane effective action. Now, the entire theory consists of the theory on the branes coupled to the bulk ten dimensional fields. We, therefore, conclude that the bulk theory must have an $SO(4)$ variation that cancels that of the brane theory. The details of this have been worked out in various examples [43, 44] (although not explicitly for the D1-D5 system, to our knowledge), and we will examine this in more detail in our next example. One finds that there is an inflow of current from the bulk region onto the branes that reproduces the anomalous divergence of the brane current. Here, we will just take for granted

[8] There is some choice in the form of the right-hand side due to the freedom to add non-gauge invariant local counterterms to the action.

that such a cancelation indeed takes place for the D1-D5 system coupled to the IIB theory.

We now turn to the supergravity description of the D1-D5 system. The starting point is the action of IIB supergravity. We will only write down the dependence on the metric, dilaton, and RR 3-form field strength, since the other fields will be vanishing in the solution. In particular, this means that we need not concern ourselves with the subtleties associated with the self-duality of the 5-form field strength.

$$I = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{G}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 + \frac{1}{2}e^{2\Phi}|G_3|^2\right). \tag{92}$$

The action is written in Euclidean signature and in terms of the string frame metric.

The equations of motion admit the following solution representing the D1-D5 system,

$$\begin{aligned} ds^2 &= (Z_1Z_5)^{-1/2}(dt^2 + dx_5^2) + (Z_1Z_5)^{1/2}dx^i dx^i + (Z_1/Z_5)^{1/2}ds_{M_4}^2 \\ G_3 &= 2Q_5\epsilon_3 + 2iQ_1 e^{-2\Phi} \star_6 \epsilon_3 \\ e^{-2\Phi} &= Z_5/Z_1. \end{aligned} \tag{93}$$

Here ϵ_3 is the volume form on the unit 3-sphere, and $\star_6$ is the Hodge dual in six dimensions. The branes intersect over (t, x_5), and we denote the four noncompact spatial direction by x^i. The harmonic functions $Z_{1,5}$ are

$$Z_{1,5} = 1 + \frac{Q_{1,5}}{r^2}, \quad Q_1 = \frac{(2\pi)^4 g N_1 \alpha'^3}{V_4}, \quad Q_5 = gN_5\alpha'. \tag{94}$$

To isolate the near horizon geometry we drop the 1 from the harmonic functions, and arrive at

$$\begin{aligned} ds^2 &= \frac{r^2}{\ell^2}(dt^2 + dx_5^2) + \frac{\ell^2}{r^2}dr^2 + \ell^2 d\Omega_3^2 + (Q_1/Q_5)^{1/2}ds_{M_4}^2 \\ G_3 &= 2Q_5(\epsilon_3 + i\star_6\epsilon_3) \\ e^{-2\Phi} &= Q_5/Q_1. \end{aligned} \tag{95}$$

with

$$\ell^2 = (Q_1Q_5)^{1/2}. \tag{96}$$

A change of coordinates brings the (t, x_5, r) part of the metric to the form (4),[9] and so we recognize the geometry as $AdS_3 \times S^3 \times M_4$.

According to the Brown-Henneaux formula (20) the central charge is given by $c = \frac{3\ell}{2G}$. Here G refers to the three dimensional Newton's constant, which we compute as

[9] Actually, since in our case x_5 is compact we only get (4) locally. To get precisely (4) one should instead start with a rotating version of the D1-D5 metric [57, 58, 59, 60].

$$G = G_{10}\frac{1}{V_7} = 8\pi^6\alpha'^4 g^2 e^{2\Phi}\frac{1}{2\pi^2\ell^3(Q_1/Q_5)V_4}. \tag{97}$$

Therefore

$$c = 6N_1N_5, \tag{98}$$

in agreement with the microscopic result.

We now ask *why* the microscopic and supergravity computations of the central charges agree. Underlying the agreement is the conjectured AdS/CFT duality between the two descriptions, but it is more satisfying to give a direct argument using just known facts. The key to this will be the relation between c and k, as well as the anomaly inflow mechanism. Compare the weak coupling description of our system, as a bound state of branes sitting in an ambient flat spacetime, to the supergravity description, as a near horizon AdS geometry joined to an asymptotically flat region. We can think of interpolating between these two pictures by dialing the string coupling. In the brane description we convinced ourselves that the chiral anomaly of the brane theory was canceled by the bulk theory via an inflow of current. Now consider what happens as we increase the string coupling. The brane anomaly is fixed in terms of the integer k and hence does not change, implying also that the bulk inflow is unchanged as we increase the coupling. But in the supergravity description we join this same asymptotic bulk geometry onto the near horizon region. In the supergravity picture, we separate the two regions by an artificial border. It is clear that the current flows smoothly across the border, and there are no sources or sinks of currents there. We conclude that the current inflow from the asymptotic region must precisely equal that into the near horizon region. But now, putting things together, we have found that the anomaly of the brane theory must precisely match the AdS current inflow. Since both are determined by a parameter k, it must be the case that $k_{CFT} = k_{sugra}$, which is what we were trying to establish. To summarize, the matching of k, and hence c, between the CFT and gravity descriptions is dictated by anomaly cancelation. A mismatch between the two would imply that IIB string theory is anomalous, which we know not to be the case.

This does not yet explain why our gravity computation yielding (98) agrees *exactly* with the CFT result, since we started from the two-derivative approximation to the supergravity action. Our anomaly argument really applies to the full action, with all higher derivative terms included. But in general, instead of using $c = 3\ell/2G$, we should instead use (36), which in general receives correction from higher derivative terms. However, the spirit of the anomaly argument suggests that we are better off computing k. We will now outline this computation.

To compute k we need to find the coefficient of the Chern-Simons term (83). The importance of this term is that under a gauge transformation, $\delta A = d\Lambda + [\Lambda, A]$, we have

$$\delta I_{CS} = -\frac{ik}{4\pi}\int_{\partial AdS}\mathrm{Tr}(\Lambda dA). \tag{99}$$

This is the result for one of the $SU(2)$ factors of the $SO(4)$ gauge group; the other factor gives the same result except with a flipped sign. Our strategy is then to similarly compute the gauge variation of (92) and compare this to (99) to read off k.

This turns out to be more subtle than one might have expected. We will sketch the main points, directing the reader to [45] for a more thorough treatment. The first step is to identify the $SO(4)$ gauge fields, which we recall are the Kaluza-Klein gauge fields associated with rotations of the S^3. Instead of the pure $AdS_3 \times S^3 \times M_4$ metric in (95) we make the replacement

$$d\Omega_3^2 \;\rightarrow\; (dy^i - A^{ij}y^j)(dy^i - A^{ik}y^k), \tag{100}$$

with $y^{1,2,3,4}$ obeying $\sum_{i=1}^4 y^i y^i = 1$. The 1-forms $A^{ij} = -A^{ji}$ are the KK gauge fields, and are allowed to have dependence on the AdS coordinates (t, x_5, r). To be more precise, we just impose this form of the metric asymptotically as we approach the AdS boundary. Note that the metric is invariant under

$$A^{ij} \rightarrow d\Lambda^{ij} + [\Lambda, A]^{ij}, \quad y^i \rightarrow y^i + \Lambda^{ij} y^j. \tag{101}$$

Given this asymptotic form of the metric, we also need to specify the asymptotic form of G_3, generalizing that in (95). This is where the subtlety lies. We need G_3 to be closed and to have a fixed integral over S^3, since this is the D5-brane charge. We might also try to have G_3 invariant under (101). If so, we would then find that the action is gauge invariant (i.e. invariant under $\delta A = d\Lambda + [\Lambda, A]$), as follows from the diffeomorphism invariance of (92) combined with the invariance under (101). It turns out that it is not possible to satisfy all these conditions simultaneously, and the best one can do is to find a G_3 that varies under (101) as

$$\delta G_3 = \frac{1}{4} Q_5 \mathrm{Tr}\left(d\Lambda dA - d\tilde{\Lambda} d\tilde{A}\right). \tag{102}$$

While this implies that the action (92) is not gauge invariant, it can be shown that the variation is a boundary term, so the equations of motion are gauge invariant. In particular, the explicit computation yields

$$\delta I = -\frac{ik}{4\pi} \int_{\partial AdS} \mathrm{Tr}\left(\Lambda dA - \tilde{\Lambda} d\tilde{A}\right), \tag{103}$$

with $k = N_1 N_5$. This then reproduces $c = 6k = 6N_1N_5$.

To complete this circle of ideas we should now show that this conclusion is unchanged under the addition of higher derivative terms to (92). This has not yet been demonstrated explicitly. Here we will just note that the condition that the action varies only by a boundary term greatly restricts the form of the action and makes it plausible that there are no further corrections. In any case, in the example discussed in the next section involving M5-branes we will give the complete derivation.

4.2 Wrapped M5-Branes

Our other example of an AdS_3 geometry arises from wrapping M5-branes on a 4-cycle in M_6, where M_6 can be T^6, $K3 \times T^2$ or CY_3. Starting from the eleven dimensional M-theory compactified on M_6, this produces a string like object in the five noncompact directions. We further compactify the direction along the string, to leave four noncompact directions. This system was studied extensively in [55, 61].

Letting Ω_I be a basis of 4-cycles in M_6, we take the M5-brane to wrap $P = p^I\Omega_I$. At low energies the theory on the resulting string flows to a CFT with (0,4) susy. The left and right moving central charges can be computed by studying the massless fluctuations of the M5-brane, including the self-dual worldvolume 3-form field strength. The result of this analysis is

$$c = C_{IJK}p^I p^J p^K + c_{2I}p^I, \quad \tilde{c} = C_{IJK}p^I p^J p^K + \frac{1}{2}c_{2I}p^I. \tag{104}$$

C_{IJK} denotes the number of triple intersections of the three 4-cycles labeled by *I, J, K* (note that three 4-cycles generically intersect over a point in six dimensions.) c_2 is the second Chern class of M_6 which we can expand in a basis of 4-forms with expansion coefficients c_{2I}. For our purposes, the main point is that C_{IJK} and c_{2I} are certain topological invariants, and so we see that the central charges are moduli independent.

Instead of studying the fluctuation problem, we can compute the central charges from the anomaly inflow mechanism [43, 44]. The relevant anomalies are those with respect to the right moving $SU(2)$ R-symmetry, and with respect to worldvolume diffeomorphisms. The relevant terms in the eleven dimensional action are

$$\begin{aligned} 2\kappa_{11}^2 I = &\int d^{11}x\sqrt{g}\left(R + \frac{1}{2}|F_4|^2\right) + \frac{i}{6}\int A_3 \wedge F_4 \wedge F_4 \\ &+ \frac{i(2\kappa_{11}^2)^{2/3}}{3\cdot 2^6\cdot(2\pi)^{10/3}}\int A_3 \wedge \left[\mathrm{Tr}R^4 - \frac{1}{4}(\mathrm{Tr}R^2)^2\right]. \end{aligned} \tag{105}$$

The terms in the first line are the standard two derivative bosonic terms. In the second line, we have written a particular eight derivative term. Of course, there are an infinite series of other higher derivative terms (see [62, 63, 64, 65, 66, 67, 68, 69, 70] for some general results), but the important point is that in (105) we have written the only two Chern-Simons terms (i.e. the only terms involving an explicit appearance of A_3). Demanding that the action be gauge invariant up to boundary terms only allows these two Chern-Simons terms, and their coefficients are fixed by a combination of supersymmetry and 1-loop computations in the dimensionally reduced IIA theory. Alternatively, the anomaly inflow computation we will now describe can be viewed as another derivation of these coefficients.

We now reduce the action to five dimensions in the presence of the M5-brane. This gives various terms, including

$$\begin{aligned} 2\kappa_5^2 I = &\int d^5x\sqrt{g}\left(R + \frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{J\mu\nu}\right) + \frac{i}{6}\int C_{IJK}A^I \wedge F^J \wedge F^K \\ &+ \frac{i\kappa_5^2}{192\pi^2}c_{2I}p_0^I\int A \wedge \mathrm{Tr}R \wedge R. \end{aligned} \tag{106}$$

Here A^I are 1-form potentials, obtained by expanding $A_3 = A^I \wedge J_I$ where J_I are a basis of (1, 1) forms. The M5-brane defines a particular magnetic charge with respect to a linear combination of gauge fields that we have called A. p_0^I defined so that $p^I = \left(-\frac{1}{2\pi}\int_{S^2} F\right) p_0^I$, where (locally) $F = dA$. G_{IJ} is a metric on the vectormultiplet moduli space, whose form can be found in, e.g, [71]. It turns out to be convenient to choose units with $\kappa_5^2 = 2\pi^2$, and so we do this from now on. This simplifies the relation between integrally conserved charges and flux integrals.

Both Chern-Simons terms in (106) contribute to the current inflow and hence to the central charges (104). By counting powers of gauge fields we can see that the *AFF* term yields the terms in the central charges cubic in the p^I, while the *ARR* term yields the linear terms.

It turns out that the cubic term is more difficult to obtain. The idea is that one needs to carefully define the action (106) in the presence of the string source, which naively acts as a delta function source. After smoothing out the source and defining the AFF term appropriately, one indeed reproduces the cubic terms in c and $\tilde{c}$. We direct the reader to [43, 44] for the analysis. We will shortly carry out a corresponding analysis in the near horizon geometry of the string, and reproduce this result in a simpler way. Right now we just emphasize that only the *AFF* Chern-Simons term is needed for the result.

Turning to the *ARR* term in the second line of (106), we now show how to compute the linear terms in the central charges. We first need to rewrite the action. Since the string is magnetically charged, A is not globally defined. The correct version of the Chern-Simons term corresponds to integrating by parts,

$$I = -\frac{ic_{2I}p_0^I}{384\pi^2}\int F \wedge \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right). \tag{107}$$

Now perform a coordinate transformation $\delta\Gamma = dv + [\Gamma, v]$,

$$\delta I = -\frac{ic_{2I}p_0^I}{384\pi^2}\int F \wedge \mathrm{Tr}(dv \wedge d\Gamma). \tag{108}$$

In the present context, we are thinking about the string as a localized source, so $dF = 0$ except at the string, where it is a delta function in the transverse space. Hence if we integrate (108) by parts, as well as integrate over the transverse space, we obtain

$$\delta I = -\frac{ic_{2I}p^I}{192\pi}\int \mathrm{Tr}(v d\Gamma). \tag{109}$$

v and Γ are 5×5 matrices, since they originated as the connection in five dimensions. We can write them in block diagonal form, corresponding to the conections on the tangent and normal bundles with respect to the string worldvolume. Taking v to act on the tangent space, the variation (109) corresponds to a gravitational anomaly, and by comparing with (88) and (89) we read off

$$c - \tilde{c} = \frac{1}{2}c_{2I}p^I. \tag{110}$$

Now taking v to act in the transverse space we obtain the "normal bundle anomaly". From the worldvolume point of view this is the same as the $SU(2)$ R-symmetry anomaly. Relabeling in $SU(2)$ language: $\Gamma^{ab} \to \tilde{A}^{ab} = \epsilon^{abc}\tilde{A}^c$ (and similarly relabeling v as $\tilde{\Lambda}$) the variation becomes

$$\delta I = \frac{ic_{2I}p^I}{48\pi}\int \mathrm{Tr}(\tilde{\Lambda} d\tilde{A}). \tag{111}$$

Comparing with (103) we read off $\tilde{k} = c_{2I}p^I/12$, or $\tilde{c} = \frac{1}{2}c_{2I}p^I$. We remind the reader that this is just the contribution linear in p^I. Combining this result with (110), we correctly reproduce the linear terms in (104).

The success of the anomaly inflow computation in reproducing the microscopic central charges can be thought of as a consistency check. Any mismatch would imply that M-theory is quantum mechanically inconsistent in the presence of M5-branes. From a practical standpoint, if we accept that anomalies should cancel, the inflow method is a very efficient means of extracting the central charges, since we only need to know the Chern-Simons terms in the effective action, and these are highly constrained.

We now shift gears and turn to the analysis in the near horizon region. The asymptotically flat solution of the five dimensional theory (106) is

$$\begin{aligned} ds^2 &= \left(\frac{1}{6}C_{IJK}H^I H^J H^K\right)^{-1/3}\left(-dt^2 + dx_4^2\right) + \left(\frac{1}{6}C_{IJK}H^I H^J H^K\right)^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right) \\ A^I &= \frac{1}{2}p^I(1+\cos\theta)\,d\phi \\ H^I &= \overline{X}^I + \frac{p^I}{2r}. \end{aligned} \tag{112}$$

The vectormultiplet moduli also take nontrivial values that we have not written out. See, e.g., [13].

To examine the near horizon geometry we write

$$r = \frac{\frac{1}{6}C_{IJK}p^I p^J p^K}{2z^2}. \tag{113}$$

For $z \to \infty$ we then find the following $\mathrm{AdS}_3 \times S^2$ geometry

$$ds^2 = \ell^2\frac{-dt^2 + dx_4^2 + dz^2}{z^2} + \frac{1}{4}\ell^2 d\Omega_2^2 \tag{114}$$

with

$$\ell = \left(\frac{1}{6}C_{IJK}p^I p^J p^K\right)^{1/3}. \tag{115}$$

The Brown-Henneaux computation of the central charge applied to this case gives (recalling $G_5 = \kappa_5^2/8\pi = \pi/4$)

$$c = \frac{3\ell}{2G_3} = \frac{3\pi\ell^3}{2G_5} = C_{IJK}p^I p^J p^K. \tag{116}$$

This result, along with the form of the solution, can alternatively be derived by the method of c-extremization as described earlier.

As expected, (116) yields the leading large charge contribution to the central charge. We now turn to the AdS computation of the exact central charges using anomalies. As we have discussed, this reduces to determining the exact coefficients of the gauge and gravitational Chern-Simons terms in the three dimensional effective action. That is, given

$$I_{CS} = \frac{i\tilde{k}}{4\pi}\int \mathrm{Tr}\left(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3\right) - i\beta\int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) \tag{117}$$

we can read off

$$\tilde{c} = 6\tilde{k}, \quad c - \tilde{c} = 96\pi\beta. \tag{118}$$

The three dimensional Chern-Simons terms descend from those in eleven dimensions, (105), or equivalently in five dimensions, (106). To read off the desired terms we can consider the following metric deformation of $\mathrm{AdS}_3 \times S^2$

$$ds^2 = ds^2_{AdS} + \frac{1}{4}\ell^2\left(dy^i - \tilde{A}^{ij}y^j\right)\left(dy^i - \tilde{A}^{ik}y^k\right), \tag{119}$$

with $\sum_{i=1}^3 (y^i)^2 = 1$. This identifies the 1-forms $\tilde{A}^{ij}$ as the $SO(3) \cong SU(2)$ gauge fields appearing in (117). We also need to give the 2-form potential supporting the solution. From (112) the undeformed solution has $F^I = dA^I = -\frac{1}{2}p^I\epsilon_{S^2}$, where ϵ_{S^2} is the volume form on the unit two-sphere. We want a generalization consistent with $SO(3)$ gauge invariance. Since the metric is invariant under

$$\begin{aligned} y^i &\to y^i + \Lambda^{ij}y^j \\ \tilde{A}^{ij} &\to \tilde{A}^{ij} + d\Lambda^{ij} + [\Lambda, A]^{ij}, \end{aligned} \tag{120}$$

where $\Lambda^{ij} = -\Lambda^{ji}$ depends only on the AdS coordinates, we also demand this of F^I. F^I must also be closed and have a fixed integral over the S^2 fibre, since this integral gives the 5-brane charge. The unique solution to this problem is [43, 44, 45]

$$F^I = -\frac{1}{2}p^I(4\pi e_2) \tag{121}$$

with

$$\begin{aligned} e_2 &= \frac{1}{8\pi}\epsilon_{ijk}(Dy^i Dy^j - \tilde{F}^{ij})y^k \\ Dy^i &= dy^i - \tilde{A}^{ij}y^j \\ \tilde{F}^{ij} &= d\tilde{A}^{ij} - \tilde{A}^{ik}\tilde{A}^{kj}. \end{aligned} \tag{122}$$

e_2 is known as the "global angular 2-form". The AFF term in (106) will now yield $\tilde{A}$ dependent terms. To work these out, a very useful formula is [72]

$$\int e_0^{(1)} \wedge e_2 \wedge e_2 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int \mathrm{Tr}\left(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3\right), \tag{123}$$

where the integral on the left(right) is over five(three) dimensions. $e_0^{(1)}$ is defined by writing $e_2 = de_0^{(1)}$, which can always be done locally since e_2 is closed. The AFF term then yields (recall $\kappa_5^2 = 2\pi^2$)

$$I_{CS} = \frac{i}{24\pi^2}\int C_{IJK}A^I \wedge F^J \wedge F^K = \frac{i}{24\pi}C_{IJK}p^I p^J p^K \int \mathrm{Tr}\left(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3\right). \tag{124}$$

This yields the coefficient of the Chern-Simons terms cubic in p^I. Indeed, comparing with (117) and using (118) we correctly read off the cubic terms in the central charges (104).

The linear terms come from the Chern-Simons term in the second line of (106). We can follow the same steps as led to (107). The difference is that in the near horizon geometry there is no explicit string source, but rather a smooth geometry, and so $dF = 0$ everywhere, without delta function singularities. After performing the S^2 integration, (107) splits into two terms corresponding to Chern-Simons terms for the $SO(3) \cong SU(2)$ connection, and the AdS$_3$ Christoffel connection,

$$I_{CS} = \frac{ic_{2I}p^I}{48\pi}\int \mathrm{Tr}\left(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3\right) - \frac{ic_{2I}p^I}{192\pi}\int \mathrm{Tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right). \tag{125}$$

Note that the relative factor of 4 between these two terms is purely due to our use of $SU(2)$ conventions for $\tilde{A}$. From (125) we correctly read off the linear terms in the central charges.

We again want to stress that this computation yields the *exact* central charges, and that we did not need to know the full action to carry it out. Knowledge of the Chern-Simons terms suffices, since they give us the anomalies, and supersymmetry connects these to the central charges. The fact that we found exact agreement with the microscopic central charges was explained already in terms of the anomaly inflow mechanism. The match is necessary to preserve diffeomorphism invariance of M-theory in the presence of M5-branes. Even though the agreement was guaranteed to occur, it is still satisfying to see it working in explicit detail.

Now that we have verified the exact matching of the microscopic and gravitational central charges, we know that the entropy of an uncharged black hole is given by (54) and that it matches with the CFT entropy. This matching includes subleading corrections to the Bekenstein-Hawking area law, as encoded in the corrections to the central charge. It further applies to non-BPS and non-extremal black holes. Historically, the result for the corrected entropy of the BPS black holes was first obtained in [73, 74, 75] by explicitly constructing the black hole solutions in supergravity supplemented by certain R^2 terms. Surprisingly, this gives the exact result even though R^4 and higher type terms are not incorporated. However, the method of [73, 74, 75] is not successful in capturing the corrected entropy of non-BPS and non-extremal black holes [76].

We should note that our results so far are only valid to leading order in L_0 and $\tilde{L}_0$, and furthermore does not allow for the inclusion of charge. In the remainder of these lectures, we will show to how generalize in these directions.

4.3 Small Black Holes and Heterotic Strings

We have seen from anomalies that the bulk AdS_3 theory exactly reproduces the microscopic central charges (104). Since this result is exact, it can be used even in cases where the bulk geometry is highly curved and the two-derivative approximation to the action is no longer valid. It is especially interesting to consider examples where the microscopic theory is as simple as possible, so that we have good control over the microscopic entropy counting. Such "small black holes" have been the subject of much recent discussion, e.g. [7, 8, 13, 77, 78, 79, 80, 81, 82, 83]

A good example is to consider $M_6 = K3 \times T^2$ and to wrap the M5-brane on $K3$. In this case only a single magnetic charge p^I is nonzero, and hence $C_{IJK} p^I p^J p^K = 0$. This implies that in the two derivative approximation, where $c = \frac{3\ell}{2G}$, the size of the AdS_3 geometry shrinks to zero. However, from (104) we see that including higher derivatives yields $c = 24p$ and $\tilde{c} = 12p$, where we used $c_2(K3) = 24$. Strictly speaking, our supergravity analysis tells us that *if* there is a finite size AdS_3 geometry, then its central charges are as stated. To actually demonstrate the existence of the geometry require more detailed consideration of the explicit supergravity equations of motion, including higher derivatives. The state-of-the-art at the moment is to include just the supersymmetric completion of certain R^2 terms, and to show that a stabilized geometry indeed results [73, 74, 75]. While working out the precise solution is an important challenge, we would like to emphasize that getting the central charges right is not too dependent on the details, since symmetries and anomalies are enough to determine them.

The connection with the heterotic string is obtained by using heterotic/IIA duality. This duality interchanges the M5-brane (NS5-brane in the IIA language) with an elementary heterotic string. The magnetic charge p becomes the winding number of the heterotic string around an S^1. The 24 leftmoving transverse bosonic oscillators of a heterotic string yield $c = 12$; and the 8 rightmoving transverse bosonic and fermionic oscillators yield $\tilde{c} = 12$. Taking into account the winding number, we see precise agreement with the supergravity side. From our discussion so far, this means that we will find agreement in the entropies from the Cardy formula (54). Note that this agreement pertains even for non-supersymmetric and nonextremal states (both left and right movers excited).

5 Partition Functions and Elliptic Genera

So far we have discussed black hole entropy at the level of the Cardy formula. We now try to go further in establishing the AdS/CFT relation

$$Z_{AdS} = Z_{CFT}. \tag{126}$$

In this section, we discuss the definitions and properties of the CFT partition functions that we will subsequently aim to reproduce from AdS.

In full generality, we can imagine defining Z_{CFT} by tracing over the CFT Hilbert space weighted by $e^{-\beta H}$ and an arbitrary string of operators. In principle such an object has a dual AdS definition, but in practice it will be intractable to actually compute. Rather than including all possible operators, it is more tractable to just focus on conserved charges, since these are more easily identifiable on the gravity side. If we define the CFT on a circle, the two most obvious conserved charges are energy and momentum, related to the Virasoro charges as[10]

$$H = L_0 - \frac{c}{24} + \tilde{L}_0 - \frac{\tilde{c}}{24}, \quad P = L_0 - \tilde{L}_0. \tag{127}$$

The most basic partition function is thus

$$Z = \mathrm{Tr}\left[e^{-\beta H + i\mu P}\right] = e^{\frac{i\mu(c-\tilde{c})}{24}}\,\mathrm{Tr}\left[e^{2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\bar{\tau}(\tilde{L}_0 - \frac{\tilde{c}}{24})}\right], \tag{128}$$

with $\tau = (\mu + i\beta)/2\pi$. If fermions are present we also need to specify their periodicity around the circle.

Now suppose that our CFT also has conserved currents, J^I and $\tilde{J}^I$. Although we use the same index I for both, the number of left moving (holomorphic) currents J^I is independent of the number of right moving (anti-holomorphic) currents $\tilde{J}^I$. We can generalize our partition function by adding chemical potentials for the corresponding conserved charges q^I and $\tilde{q}^I$,

$$Z = \mathrm{Tr}\left[e^{2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\bar{\tau}(\tilde{L}_0 - \frac{\tilde{c}}{24})} e^{2\pi i z_I q^I} e^{-2\pi i \tilde{z}_I \tilde{q}^I}\right]. \tag{129}$$

The path integral version of the partition function (5.4) is,

$$Z_{PI} = \int [\mathscr{D}\Phi]\; e^{-I - \frac{i}{2\pi}\int (A^\mu J_\mu + \tilde{A}^\mu \tilde{J}_\mu)}, \tag{130}$$

where the CFT is defined on the torus. The external gauge fields appearing in (130) are related to the chemical potentials in (129),

$$z_I = -i\tau_2 A_{I\overline{w}}, \quad \tilde{z}_I = i\tau_2 \tilde{A}_{Iw}, \tag{131}$$

where $\tau = \tau_1 + i\tau_2$. Further, the the path integral and canonical versions are related as

$$Z = e^{-\frac{\pi}{\tau}(z^2 + \tilde{z}^2)} Z_{PI}, \tag{132}$$

with $z^2 = k^{IJ} z_I z_J$, (k^{IJ} is defined in (78)), and similarly for $\tilde{z}^2$.

[10] P is the same as what we earlier called J, the AdS angular momentum.

To derive (131) and (132), it is most instructive to consider a simple example of a free scalar field. This example will also allow us to discuss the modular behavior of our partition functions.

5.1 Free Scalar Field Example

Consider a free compact boson of radius $2\pi R$. We use the conventions of [84] and set $\alpha' = 1$. We define the partition function

$$Z(\tau, z, \tilde{z}) = (q\bar{q})^{-1/24} \mathrm{Tr}\left[q^{L_0} \bar{q}^{\tilde{L}_0} e^{2\pi i z p_L} e^{2\pi i \tilde{z} p_R} \right], \tag{133}$$

with

$$\begin{aligned} L_0 &= \frac{p_L^2}{4} + L_0^{osc}, \quad \tilde{L}_0 = \frac{p_R^2}{4} + \tilde{L}_0^{osc} \\ p_L &= \frac{n}{R} + wR, \quad p_R = \frac{n}{R} - wR. \end{aligned} \tag{134}$$

The partition function obeys the modular transformation rule

$$Z\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \frac{\tilde{z}}{c\bar{\tau} + d} \right) = e^{\frac{2\pi i c z^2}{c\tau + d}} e^{-\frac{2\pi i c \tilde{z}^2}{c\bar{\tau} + d}} Z(\tau, z, \tilde{z}), \tag{135}$$

as is readily verified by direct computation.

To explain the origin of the exponential prefactors in (135) we pass to a path integral formulation. We consider

$$Z_{PI}(\tau, A) = \int \mathscr{D}X e^{-I} \tag{136}$$

with

$$I = \frac{1}{2\pi} \int_{T^2} d^2\sigma \sqrt{g} \left[\frac{1}{2} g^{ij} \partial_i X \partial_j X - A^i \partial_i X \right] \tag{137}$$

and $A^i =$ constant. To relate potentials appearing in (133) and (134), we use the standard expression for the charges

$$p_L = 2 \oint \frac{dw}{2\pi i} i\partial_w X, \quad p_R = -2 \oint \frac{dw}{2\pi i} i\partial_{\overline{w}} X, \tag{138}$$

and then equate the charge-dependent phases in the two versions. This yields

$$z = -i\tau_2 A_{\overline{w}}, \quad \tilde{z} = i\tau_2 \tilde{A}_w. \tag{139}$$

We denoted the holomorphic part of the gauge field $\tilde{A}_w$ because, elsewhere in these notes, this component arises from an independent bulk 1-form $\tilde{A}$.

In the path integral formulation a modular transformation is a coordinate transformation combined with a Weyl transformation, and so it is manifest that

$$Z_{PI}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},\frac{\tilde{z}}{c\bar{\tau}+d}\right)=Z_{PI}(\tau,z,\tilde{z}),\tag{140}$$

where the transformation of z and $\tilde{z}$ just expresses the coordinate transformation.

What then is the relation between Z_{PI} and Z? To find this, we just carry out the usual steps that relate Hamiltonian and path integral expressions (e.g. $\int \mathscr{D}Xe^{-I} = \mathrm{Tr}e^{-\beta H}$.) The only point to be aware of is that the Hamiltonian corresponding to the action (137) is not the factor appearing in the exponential of (133) but differs from this by a contribution quadratic in the potentials, as is verified by carrying out the standard Legendre transformation. In particular, we find

$$Z_{PI}(\tau,z,\tilde{z})=e^{\frac{\pi(z+\tilde{z})^2}{\tau_2}}Z(\tau,z,\tilde{z}).\tag{141}$$

Combining (140) and (141) we see that the modular transformation law of Z must be such to precisely offset that of $e^{\frac{\pi(z+\tilde{z})^2}{\tau_2}}$. This is what (135) does.

To summarize, we have shown how to convert between the canonical and path integral versions of the partition function. The latter makes the modular behavior manifest. Furthermore, the analysis we performed is essentially completely general, in that given an arbitrary CFT we can always realize the $U(1)$ current algebra in terms of free bosons.

5.2 Elliptic Genus

The partition function (129) receives contributions from all states of the theory. This makes it intractable to calculate explicitly, except in favorable cases (such as weak coupling limits). In a theory with enough supersymmetry, we can define a more controlled object – the "elliptic genus" – which only receives contributions from BPS states. The elliptic genus is a topological invariant, as we will review in a moment, which allows it to be computed far more readily than the generic partition function. Useful references include [10, 85, 86].

For definiteness, we now focus on a CFT with (0,4) susy. The elliptic genus is defined as

$$\chi(\tau,z_I)=\mathrm{Tr}_R\left[e^{2\pi i\tau(L_0-c/24)}e^{-2\pi i\bar{\tau}(\tilde{L}_0-\tilde{c}/24)}e^{2\pi i z_I q^I}(-1)^{\tilde{F}}\right].\tag{142}$$

The trace is over the Ramond sector, and $\tilde{F}$ is the fermion number, defined as $\tilde{F}=2\tilde{J}_0^3$, where $\tilde{J}_0^3$ is the R-charge. The insertion of $(-1)^{\tilde{F}}$ imposes a bose-fermi cancelation among all states *except* those obeying $\tilde{L}_0-\tilde{c}/24=0$ (the Ramond ground states). The arguments here are the same as in the study of the Witten index in 4D supersymmetric field theories. Since only states with $\tilde{L}_0 - \tilde{c}/24 = 0$ contribute, the elliptic genus does not depend explicitly on $\bar{\tau}$. On the other hand, all leftmoving states can contribute. The elliptic genus is invariant under smooth deformations of the CFT. This follow from the quantization of the charges and of $L_0 - \tilde{L}_0$, together with the fact that only rightmoving ground states contribute. We can therefore compute the elliptic genus in the free limit of the CFT and then extrapolate it to strong coupling and compare with a supergravity computation.

We now state the main general properties of the elliptic genus.

5.2.1 Modular Transformation

$$\chi\left(\frac{a\tau+b}{c\tau+d}, \frac{z_I}{c\tau+d}\right) = e^{2\pi i \frac{cz^2}{c\tau+d}} \chi(\tau, z_I). \tag{143}$$

The same argument applies here as in (141).

5.2.2 Spectral Flow

The modes of the stress tensor and currents obey the algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, J_n^I] &= -nJ_{m+n}^J, \\
[J_m^I, J_n^J] &= \frac{1}{2} m k^{IJ} \delta_{m+n,0}.
\end{aligned} \tag{144}$$

This is invariant under the spectral flow automorphism (79).

The spectral flow automorphism implies the relation

$$\chi(\tau, z_I + \ell_I \tau + m_I) = e^{-2\pi i(\ell^2 \tau + 2\ell\cdot z)} \chi(\tau, z_I), \tag{145}$$

where m_I obeys $m_I q^I \in \mathbb{Z}$, and we defined $\ell^2 = k^{IJ}\ell_I \ell_J$, $\ell \cdot z = k^{IJ} \ell_I z_J$. It also implies that if we expand the elliptic genus as

$$\chi(\tau, z_I) = \sum_{n, r^I} c(n, r^I) e^{2\pi i n\tau + 2\pi i z_I r^I}, \tag{146}$$

then the expansion coefficients are a function of a single spectral flow invariant combination:

$$c(n, r^I) = c\left(n - \frac{r^2}{4}\right). \tag{147}$$

Here we defined $r^2 = k_{IJ} r^I r^J$, where k_{IJ} denotes the inverse of k^{IJ}.

5.2.3 Factorization of Dependence on Potentials

We can explicitly write the dependence of the elliptic genus on the potentials z_I. The intuition behind this is that we can always separate the CFT into the currents plus everything else, and the current part can be realized in terms of free bosons. We have:

$$\chi(\tau,z_I) = \sum_{\mu^I} h_\mu(\tau)\Theta_{\mu,k}(\tau,z_I), \tag{148}$$

with

$$\Theta_{\mu,k}(\tau,z_I) = \sum_{\eta_I} e^{\frac{i\pi\tau}{2}(\mu+2k\eta)^2} e^{2\pi i z_I(\mu^I+2k^{IJ}\eta_J)}. \tag{149}$$

We are using the shorthand notation

$$(\mu+2k\eta)^2 \equiv k_{IJ}\left(\mu^I+2k^{IK}\eta_K\right)\left(\mu^J+2k^{JL}\eta_L\right). \tag{150}$$

The combined sum over μ^I and η_I includes the complete spectrum of charges. The sum over η_I corresponds to shifts of the charges by spectral flow, and so the sum on μ_I is over a fundamental domain with respect to these shifts. A more intuitive understanding of (148), (149) will emerge when we rederive these results from the AdS side.

5.2.4 Farey Tail Expansion

The main observation of [10] was that upon applying the "Farey tail transform", the elliptic genus admits an expansion that is suggestive of a supergravity interpretation in terms of a sum over geometries. We will essentially state the result here, referring to [10] for the detailed derivation. The CFT discussion in [10] has recently been adapted to the (0,4) context in [87].

The properties (143) and (145) are the definitions of a "weak Jacobi form" of weight $w=0$ and index k. Actually, the definition strictly applies when k is a single number rather than a matrix, but we will still use this langauge.

The Farey tail transformed elliptic genus is

$$\tilde{\chi}(\tau,z_I) = \left(\frac{1}{2\pi i}\partial_\tau - \frac{1}{4}\frac{\partial_z^2}{(2\pi i)^2}\right)^{3/2}\chi(\tau,z_I), \tag{151}$$

where $\partial_z^2 = k_{IJ}\partial_{z_I}\partial_{z_J}$. $\tilde{\chi}$ is a weak Jacobi form of weight 3 and index k and admits the expansion

$$\tilde{\chi}(\tau,z_I) = e^{-\frac{\pi z^2}{\tau_2}} \sum_{\Gamma_\infty\backslash\Gamma} \frac{1}{(c\tau+d)^3}\hat{\chi}\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{ct+d}\right), \tag{152}$$

with

$$\hat{\chi}(\tau, z_I) = e^{\frac{\pi z^2}{\tau_2}} \sum_{\mu,\tilde{\mu},m,\tilde{m}}^{\hat{}} \tilde{c}\left(m, \mu^I\right) e^{2\pi i (m - \frac{1}{4}\mu^2)^{\tau}} \Theta_{\mu,k}(\tau, z_I), \tag{153}$$

and $\Theta_{\mu,k}(\tau,z_I)$ was defined in (149). The hatted summation appearing in (153) is over states with $m - \frac{1}{4}\mu^2 < 0$. From the gravitational point of view, these will be states below the black hole threshold and the sum over $\Gamma_\infty\backslash\Gamma$ then adds the black holes back in. In mathematical terminology (153) defines $\hat{\chi}$ as the "polar part" of the elliptic genus. The coefficients $\tilde{c}(m,\mu^I)$ in (153) are related to those in (146) by

$$\tilde{c}\left(m, \mu^I\right) = \left(m - \frac{\mu^2}{4}\right)^{3/2} c\left(m - \frac{\mu^2}{4}\right), \tag{154}$$

as follows from (151) and from using (147). The main point is that the transformed elliptic genus $\tilde{\chi}$ can be reconstructed in terms of its polar part $\hat{\chi}$.

6 Computation of Partition Functions in Gravity: Warmup Examples

We now turn to the gravitational computation of partition functions, particularly the elliptic genus. One goal will be to see how the general properties described in the previous section are realized in terms of the sum over geometries. For example, we need to see how a sum over black hole geometries, with the precise weighting factors specified by (152) and (153), arises in the AdS description.

Before considering the general problem of summing over geometries, it will be helpful to get oriented by considering some examples. Again, for definiteness we will focus on the (0,4) case, although the generalization to the (4,4) case is very straightforward.

6.1 NS Vacuum

The NS vacuum is invariant under $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. In other words, it is invariant under the full group of AdS_3 isometries, which means that it is precisely global AdS_3,

$$ds^2 = (1 + r^2/\ell^2)\ell^2 dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2. \tag{155}$$

The contractibility of the ϕ circle forces the fermions to be anti-periodic in ϕ. Invariance under the isometry group means that this geometry has

$$L_0 = \tilde{L}_0 = 0. \tag{156}$$

6.2 Spectral Flow to the R Sector

On the gravity side a rightmoving spectral flow (79) is implemented by a constant shift in the gauge potentials (80), but now in terms of the rightmoving tilded version. To get to the Ramond sector we want to flip the periodicity of the supercurrent. This carries charge $\tilde{q}^0 = 1$, and so we should take $\tilde{\eta}_0 = \frac{1}{2}$. Therefore, a Ramond ground states consists of the metric (155) with

$$\tilde{A}_{0\overline{w}} = 1, \tag{157}$$

with fermions periodic in ϕ. The gauge field contribution (76) increases the Virasoro charge from (156) to

$$\tilde{L}_0 = \frac{\tilde{k}}{4} = \frac{\tilde{c}}{24}. \tag{158}$$

Since the charge (77) is

$$\tilde{q}^0 = \tilde{k} = \frac{\tilde{c}}{6}, \tag{159}$$

this is the maximally charged R vacuum state.[11] To get the maximally negatively charged R vacuum we flip the sign in (157). In the (4,4) case the leftmoving side is treated analogously.

6.3 Conical Defects

A more general class of R vacua are the conical defect geometries [58, 59, 60]. For these we take

$$ds^2 = \left(\frac{1}{N^2} + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(\frac{1}{N^2} + \frac{r^2}{\ell^2}\right)} + r^2 d\phi^2, \tag{160}$$
$$\tilde{A}_{0\overline{w}} = \frac{1}{N},$$

with $N \in \mathbb{Z}$. The angular coordinate ϕ has the standard 2π periodicity, and fermions are taken to be periodic in ϕ.

To read off the Virasoro charges, we just note that by rescaling coordinates all these geometries are locally equivalent to the $N = 1$ case discussed in the previous example. In the $N = 1$ case the right moving stress tensor vanishes, and it will clearly continue to vanish after rescaling coordinates. Thus (158) still applies and so $\tilde{L}_0 = \frac{\tilde{k}}{4}$ as before. The R-charge is read off from (76) to (77) as

$$\tilde{q}^0 = \frac{\tilde{k}}{N}. \tag{161}$$

[11] This bound on the charge can be seen from the supersymmetry algebra.

Upper and lower bounds on N are given by the quantization of R-charge, so $|N| \leq \tilde{k}$.

These conical defect geometries are singular at the origin unless the holonomy is ± 1, which corresponds to $N = \pm 1$. In the context of the D1-D5 system, the singular geometries are known to be physical in that the singularity corresponds to the presence of N coincident Kaluza-Klein monopoles. Another way of viewing this is that these singular geometries are special limits of the much larger class of smooth RR vacua geometries that have been heavily studied in recent years [88, 89].

We also note that any of the R-vacua in (160) can be spectral flowed to the NS sector to give chiral primary geometries.

6.4 Black Holes

We now consider black hole geometries, and give a simple derivation of the entropy of charged black holes that incorporates higher derivative corrections. This will provide the generalization of (54). We again use the method of relating the black hole to thermal AdS by a modular transformation. We will be considering a general, rotating, non-extremal, charged black hole. All left and right moving charges will be turned on.

The starting point is global AdS_3, as in (155). The complex boundary coordinate is $w = \phi + it/\ell$, and we identify $w \cong w + 2\pi \cong w + 2\pi\tau$. To add charge we also want to turn on flat potentials for the gauge fields. Now, the ϕ circle is contractible in the bulk, so to avoid a singularity at the origin, we need to set to zero the ϕ component of all potentials. We therefore allow nonzero $A_{Iw} = -A_{I\overline{w}}$, and $\tilde{A}_{I\overline{w}} = -\tilde{A}_{Iw}$.

What is the action associated with this solution? From the discussion in Sect. 3, we know the exact expressions for the stress tensor and currents

$$\begin{aligned} T_{ww} &= -\frac{k}{8\pi} + \frac{1}{8\pi} A_w^2 + \frac{1}{8\pi} \tilde{A}_w^2, \\ T_{\overline{w}\overline{w}} &= -\frac{\tilde{k}}{8\pi} + \frac{1}{8\pi} A_{\overline{w}}^2 + \frac{1}{8\pi} \tilde{A}_{\overline{w}}^2, \\ J_w^I &= \frac{i}{2} k^{IJ} A_{Jw}, \\ \tilde{J}_{\overline{w}}^I &= \frac{i}{2} \tilde{k}^{IJ} \tilde{A}_{J\overline{w}}. \end{aligned} \tag{162}$$

To obtain the exact action from these formulae we need to integrate the equation

$$\delta I = \int_{\partial AdS} d^2x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta}^{(0)} + \frac{i}{2\pi} J^{I\alpha} \delta A_{I\alpha} + \frac{i}{2\pi} \tilde{J}^{I\alpha} \delta \tilde{A}_{I\alpha} \right). \tag{163}$$

As we did to derive (43), we first need to switch to the z coordinates (41) that have fixed periodicities. Doing this, then switching back to the w coordinates, we find

$$\delta I = (2\pi)^2 i \left[-T_{ww} \delta\tau + T_{\overline{w}\overline{w}} \delta\overline{\tau} + \frac{\tau_2}{\pi} J_w^I \delta A_{I\overline{w}} + \frac{\tau_2}{\pi} \tilde{J}_{\overline{w}}^I \delta \tilde{A}_{Iw} \right]_{\text{const}}. \tag{164}$$

The *const* subscript indicates that we keep just the zero mode part. Inserting (162) into this equation we can now integrate and find our desired action as

$$I = \frac{i\pi k}{2}\tau - \frac{i\pi\tilde{k}}{2}\overline{\tau} + \pi\tau_2\left(A_{\overline{w}}^2 + \tilde{A}_w^2\right). \tag{165}$$

A simpler derivation of this result is to just compute (165) by directly evaluating the action on the solution. The gauge field contribution just comes from the boundary terms in (75). The reason we proceeded in terms of (163) was to emphasize that the result (162) is exact for an arbitrary higher derivative action, and also because we will generalize this computation later.

The result (165) is the action for the AdS_3 ground state with a flat connection turned on. Next, we perform the modular transformation $\tau \to -1/\tau$ in order to reinterpret the solution as a Euclidean black hole. This is implemented by

$$w \to -w/\tau, \quad A_{I\overline{w}} \to -\overline{\tau}A_{I\overline{w}}, \quad \tilde{A}_{Iw} \to -\tau\tilde{A}_{Iw}. \tag{166}$$

The action is of course invariant since we are just rewriting it in new variables. Using $\overline{\tau}/\tau = 1 - 2i\tau_2/\tau$ we can present the result as

$$\begin{aligned} I &= -\frac{i\pi k}{2\tau} + \frac{i\pi\tilde{k}}{2\overline{\tau}} - \frac{2\pi i\tau_2^2 A_{\overline{w}}^2}{\tau} + \frac{2\pi i\tau_2^2\tilde{A}_{\overline{w}}^2}{\overline{\tau}} + \pi\tau_2\left(A_{\overline{w}}^2 + \tilde{A}_w^2\right) \\ &= -\frac{i\pi k}{2\tau} + \frac{i\pi\tilde{k}}{2\overline{\tau}} + \frac{2\pi i z^2}{\tau} - \frac{2\pi i\tilde{z}^2}{\overline{\tau}} - \frac{\pi}{\tau_2}\left(z^2 + \tilde{z}^2\right). \end{aligned} \tag{167}$$

This is the Euclidean action of a black hole with modular parameter τ and potentials specified by z_I and $\tilde{z}_I$.

Our result (167) is the leading saddle point contribution to the path integral. As we noted in (132) the canonical form of the partition function, defined as a trace, is related to the path integral as

$$Z = e^{-\frac{\pi}{\tau_2}(z^2+\tilde{z}^2)} Z_{PI} = e^{-\frac{\pi}{\tau_2}(z^2+\tilde{z}^2)} \sum e^{-I}. \tag{168}$$

The exponential prefactor cancels the last term in (167) so that

$$\ln Z = \frac{i\pi k}{2\tau} - \frac{i\pi\tilde{k}}{2\overline{\tau}} - \frac{2\pi i z^2}{\tau} + \frac{2\pi i\tilde{z}^2}{\overline{\tau}}, \tag{169}$$

on the saddle point. We define the entropy s by writing the partition function as

$$Z = e^s e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\overline{\tau}(\tilde{L}_0 - \tilde{c}/24)} e^{2\pi i z_I q^I} e^{-2\pi i\tilde{z}_I q^I}, \tag{170}$$

where we assume that Z is dominated by a single charge configuration with, e.g., $q^I = \frac{1}{2\pi i}\frac{\partial}{\partial z_I}\ln Z$.

Putting everything together we read off the black hole entropy as

$$s = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24} - \frac{1}{4}q^2\right)} + 2\pi\sqrt{\frac{\tilde{c}}{6}\left(\tilde{L}_0 - \frac{\tilde{c}}{24} - \frac{1}{4}\tilde{q}^2\right)}. \tag{171}$$

The expression (171) gives the entropy for a general nonextremal, rotating, charged, black hole in AdS_3, including the effect of higher derivative corrections as incorporated in the central charges. Since we used the saddle point approximation the formula is only valid to leading order in $L_0 - \frac{c}{24} - \frac{1}{4}q^2$ (and the rightmoving analogue), including the subleading contribution is the topic of the next section. It is striking that we have control over higher derivative corrections to the entropy even for nonsupersymmetric black holes.[12] As in our discussion of the uncharged case, the relation with anomalies implies that (171) is in precise agreement with the microscopic entropy counting coming from brane constructions.

7 Computation of Partition Functions in Supergravity

Let us now look at the supergravity computation of the elliptic genus. We'll consider both the canonical and path integral approaches, which are useful for making manifest the behavior under spectral flow and modular transformation, respectively. In keeping with the Farey tail philosophy [10], we first explicitly compute the contribution to the elliptic genus from states below the black hole threshold. With this in hand, we then note that black holes are readily included since they are just coordinate transformations of solutions below the threshold. In this way we reproduce the construction (152).

7.1 Canonical Approach

In the canonical approach we need to enumerate the allowed set of bulk solutions and their charge assignments. For the elliptic genus, we consider states of the form (anything, R-ground state), which have $\tilde{L}_0 = \frac{\tilde{k}}{4}$. There are three classes of such states: smooth solutions in the effective three dimensional theory; states coming from Kaluza-Klein reduction of the higher dimensional supergravity theory; and non-supergravity string/brane states. Some members of the first class were discussed above, and we will make a few comments on the other types of states later.

Just as was done on the CFT side (148), it is useful to factorize the dependence on the potentials. In the gravitational context it is manifest that the stress tensor consists of a metric part plus a gauge field part. Suppose we are given a state carrying leftmoving charges

$$\left(L_0 - \frac{c}{24}, q^I\right) = \left(m, \mu^I\right). \tag{172}$$

We can apply spectral flow to generate the family of states with charges

[12] A related observation is that the attractor mechanism, which plays an important role in establishing a near horizon AdS_3 geometry, can also operate for non-supersymmetric black holes [90].

$$L_0 - \frac{c}{24} = m + \eta_I q^I + k^{IJ}\eta_I \eta_J = m - \frac{1}{4}\mu^2 + \frac{1}{4}(\mu + 2k\eta)^2$$
$$q^I = \mu^I + 2k^{IJ}\eta_J, \tag{173}$$

where we are using the same shorthand notation as in (150). This class of states will then contribute to the elliptic genus as

$$\chi(\tau, z_I) = (-1)^{\tilde{F}} e^{2\pi i \tau (m - \frac{1}{4}\mu^2)} \Theta_{\mu,k}(\tau, z_I), \tag{174}$$

in terms of the Θ-function (149). Each such spectral flow orbit has a certain degeneracy from the number distinct states with these charges. We call this degeneracy $c(m - \frac{1}{4}\mu^2)$, where the functional dependence is fixed by the spectral flow invariance, and we also include $(-1)^{\tilde{F}}$ in the definition. We can now write down the "polar" part of the elliptic genus, that is, the contribution below the black hole threshold: $m - \frac{1}{4}\mu^2 < 0$. We then have

$$\chi'(\tau, z_I) = \sum\nolimits'_{m,\mu} c\left(m - \frac{1}{4}\mu^2\right) \Theta_{\mu,k}(\tau, z_I) e^{2\pi i \left(m - \frac{1}{4}\mu^2\right)\tau}. \tag{175}$$

In the canonical approach, it is easy to write down the polar part of the elliptic genus in terms of the degeneracies $c(m - \frac{1}{4}\mu^2)$. But the full elliptic genus also has a contribution from black holes, and these are not easily incorporated since black holes do not correspond to individual states of the theory. To incorporate black holes, we need to turn to a Euclidean path integral, as we do now.

7.2 Path Integral Approach

In the path integral approach, we sum over bulk solutions with fixed boundary conditions

$$\chi_{PI}(\tau, z_I) = \sum e^{-I}. \tag{176}$$

The action appearing in (176) is the full string/M-theory effective action reduced to AdS_3, though we fortunately do not require its explicit form to compute the elliptic genus. In particular, in (176) we only sum over stationary points of I, since the fluctuations have already been incorporated through higher derivative corrections to the action.

The boundary conditions on the metric are that the boundary geometry is a torus of modular parameter τ. z_I fix the boundary conditions for the gauge potentials. As derived in (139), the relation is, in conformal gauge,

$$A_{I\overline{w}} = \frac{i z_I}{\tau_2}. \tag{177}$$

A_{Iw} is not fixed as a boundary condition. Since the potential $\tilde{z}_I$ is set to zero in the elliptic genus, we also have the boundary condition

$$\tilde{A}_{Iw} = 0. \tag{178}$$

Now we turn to the allowed values of A_{Iw} and $\tilde{A}_{I\overline{w}}$. The allowed boundary values of A_{Iw} are determined from the holonomies around the contractible cycle of the AdS$_3$ geometry. Recall that when we write $w = \sigma_1 + i\sigma_2$ we are taking σ_1 to be the 2π periodic spatial angular coordinate. The corresponding cycle on the boundary torus is contractible in the bulk, and so any nonzero holonomy must match onto an appropriate source in order to be physical. The holonomy of a charge q^I particle is

$$e^{\frac{1}{2}iq^I \int d\sigma_1 A_{I\sigma_1}} = e^{\frac{1}{2}iq^I \int d\sigma_1 (A_{Iw}+A_{I\overline{w}})}. \tag{179}$$

Choosing a gauge with constant A_{Iw}, we write the allowed values as

$$A_{Iw} = k_{IJ}\mu^J + 2\eta_I - \frac{iz_I}{\tau_2}, \quad q^I\eta_I \in \mathbb{Z}, \tag{180}$$

where we have written the charge of the source as μ^I.

In the same way, we can determine the allowed values of $\tilde{A}_{I\overline{w}}$. In this case we know that only geometries with $\tilde{L}_0 - \frac{\tilde{c}}{24} = 0$ contribute to the elliptic genus, and so we do not include the spectral flowed geometries as we did above. Instead, we just have

$$\tilde{A}_{I\overline{w}} = \tilde{k}_{IJ}\tilde{\mu}^J. \tag{181}$$

Given the gauge fields, we know the exact stress tensor and also the exact currents. We can therefore find the action by integrating

$$\begin{aligned}\delta I &= \int_{\partial AdS} d^2x\sqrt{g^{(0)}}\left(\frac{1}{2}T^{\alpha\beta}\delta g^{(0)}_{\alpha\beta} + \frac{i}{2\pi}J^{I\alpha}\delta A_{I\alpha}\right)\\ &= (2\pi)^2 i\left[-T_{ww}\delta\tau + T_{\overline{ww}}\delta\overline{\tau} + \frac{\tau_2}{\pi}J^I_w\delta A_{I\overline{w}} + \frac{\tau_2}{\pi}\tilde{J}^I_{\overline{w}}\delta\tilde{A}_{Iw}\right]_{\text{const}},\end{aligned} \tag{182}$$

as in Sect. 4.4. The result is

$$\begin{aligned}I &= -2\pi i\tau\left(L_0^{grav} - \frac{c}{24}\right) + 2\pi i\overline{\tau}\left(\tilde{L}_0^{grav} - \frac{\tilde{c}}{24}\right)\\ &\quad - \frac{i\pi}{2}\left[\tau A_w^2 + \overline{\tau}A_{\overline{w}}^2 + 2\overline{\tau}A_wA_{\overline{w}}\right] + \frac{i\pi}{2}\left[\tau\tilde{A}_w^2 + \overline{\tau}\tilde{A}_{\overline{w}}^2 + 2\tau\tilde{A}_w\tilde{A}_{\overline{w}}\right].\end{aligned} \tag{183}$$

In verifying that (183) satisfies (182) one has to take care to consider only variations consistent with the equations of motion and the assumed boundary conditions. We maintain fixed holonomies by taking $\delta A_{Iw} = -\delta A_{I\overline{w}}$ and $\delta\tilde{A}_{Iw} = -\delta\tilde{A}_{I\overline{w}}$. Also, the variation of the complex structure must be taken with the gauge field fixed in the z-coordinates introduced in (41).

The result (183) for the action agrees with (162) when the geometry is in the ground state where $A_{Iw} = -A_{I\overline{w}}$ and $\tilde{A}_{Iw} = -\tilde{A}_{I\overline{w}}$, but it is valid also more generally in the presence of charged sources. In fact, it is equivalent to the canonical result discussed in Sect. 7.1. To see this, we consider again the charge assignments (172). Writing $L_0 = L_0^{grav} + L_0^{gauge} = L_0^{grav} + \frac{1}{4}\mu^2$ (and analogously for $\tilde{L}_0$) we insert into (183) and find

$$I = -2\pi i\tau\left(m-\frac{1}{4}\mu^2\right) - \frac{i\pi\tau}{2}(\mu+2k\eta)^2 - 2\pi i z_I\left(\mu^I + 2k^{IJ}\eta_J\right) - \frac{\pi z^2}{\tau_2}. \quad (184)$$

Summing over the geometries below the black hole threshold we find

$$\begin{aligned}\chi'_{PI}(\tau,z_I) &= \sum\nolimits'_{m,\mu} c\left(m-\frac{1}{4}\mu^2\right)e^{-S} \\ &= e^{\frac{\pi z^2}{\tau_2}}\sum\nolimits'_{m,\mu} c\left(m-\frac{1}{4}\mu^2\right)\Theta_{\mu,k}(\tau,z_I)e^{2\pi i(m-\frac{1}{4}\mu^2)\tau} \\ &= e^{\frac{\pi z^2}{\tau_2}}\chi'(\tau,z_I),\end{aligned} \quad (185)$$

where χ' is the canonical result (175). As in (132), the overall exponential factor is precisely the one we expect.

7.3 Including Black Holes

Black holes are readily included in the path integral approach since they are just rewritten versions of solutions below the black hole threshold. Taking a solution below the black threshold and performing the coordinate transformation $w \to \frac{aw+b}{cw+d}$ generates a black hole. Using the manifest invariance of the action under such coordinate transformations, the contribution of such a black is then

$$\chi_{PI}(\tau,z_I) = \chi'_{PI}\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{c\tau+d}\right). \quad (186)$$

On the other hand, from the relation (185) between χ'_{PI} and χ' we have

$$\chi'_{PI}\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{c\tau+d}\right) = e^{-2\pi i\frac{cz^2}{c\tau+d}}e^{\frac{\pi z^2}{\tau_2}}\chi'\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{c\tau+d}\right). \quad (187)$$

Thus the black hole contribution to χ is

$$\chi(\tau,z_I) = e^{-\frac{\pi z^2}{\tau_2}}\chi_{PI}(\tau,z_I) = e^{-2\pi i\frac{cz^2}{c\tau+d}}\chi'\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{c\tau+d}\right). \quad (188)$$

The next step is to sum over all inequivalent black holes to get the complete elliptic genus. This means summing over the subgroup of $\Gamma = SL(2,\mathbb{Z})$ corresponding to inequivalent black holes or, more precisely, distinct ways of labeling the contractible cycle in terms of time and space coordinates. As explained in [10] the inequivalent cycles are parameterized by $\Gamma_\infty\backslash\Gamma$; so it seems natural to write

$$\chi(\tau,z_I) = \sum_{\Gamma_\infty\backslash\Gamma} e^{-2\pi i\frac{cz^2}{c\tau+d}}\chi'\left(\frac{a\tau+b}{c\tau+d},\frac{z_I}{c\tau+d}\right). \quad (189)$$

However, as emphasized in [10], this cannot be correct since the sum is not convergent. Instead we should compute not the elliptic genus but instead its Farey transform, introduced in (151). This amounts to first replacing χ' by

$$\hat{\chi}'(\tau, z_I) = \sum\nolimits'_{m,\mu} \tilde{c}\left(m - \frac{1}{4}\mu^2\right) \Theta_{\mu,k}(\tau, z_I)\, e^{2\pi i\left(m-\frac{1}{4}\mu^2\right)\tau} \tag{190}$$

with $\tilde{c}$ defined as in (154). We interpret this as the polar part of a weak Jacobi form of weight 3 and index k. Instead of (189) we therefore write

$$\hat{\chi}(\tau, z_I) = \sum_{\Gamma_\infty \backslash \Gamma} (c\tau + d)^{-3} e^{-2\pi i \frac{cz^2}{c\tau+d}} \hat{\chi}'\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right). \tag{191}$$

7.4 High Temperature Behavior

The high temperature $(\tau_2 \to 0)$ behavior of (191) is governed by the free energy of a BPS black hole. The leading exponential behavior can be read off from the term

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad m = 0, \quad \eta_I = 0, \quad \mu^I = k\delta^{I0}, \tag{192}$$

which gives

$$\hat{\chi}(\tau, z_I) \approx e^{-\frac{2\pi i z^2}{\tau} + \frac{2\pi i k z_0}{\tau}}. \tag{193}$$

We can compare with (169) by performing the spectral flow $z_0 \to z_0 + \frac{1}{2}$. This yields

$$\ln \hat{\chi}(\tau, z_I) \approx \frac{i\pi k}{2\tau} - \frac{2\pi i z^2}{\tau}. \tag{194}$$

Noting that this agrees with the holomorphic part of (169), we find that the corresponding entropy is is indeed that of a BPS black hole,

$$s = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24} - \frac{1}{4}q^2\right)}. \tag{195}$$

This is just the leading part of the entropy and is insensitive to the distinction between the elliptic genus and its Farey-tail transformed version.

7.5 Summary

It is now helpful to summarize what has been achieved so far. In our CFT discussion we noted that the CFT elliptic genus[13] is completely determined by the spectrum of

[13] Here when we say elliptic genus, we really mean its Farey tail transform.

BPS states below the black hole threshold, and by the algebra of CFT currents. By evaluating the Euclidean path integral, we then showed that the AdS elliptic genus has precisely the same structure. Thus we have boiled the question of exact agreement of the elliptic genera to the comparison of current algebras and BPS states below the black hole threshold. To complete the computation these need to be worked out. Some aspects of this problem on the AdS side are the subject of the next section.

8 Computation of BPS Spectra

There are in general two types of BPS states to consider: supergravity states from the Kaluza-Klein fluctuation spectrum of the higher dimensional theory reduced to AdS_3; and branes wrapping cycles of the internal compactification manifold. We do not intend to give a full description of either here and restrict ourselves to sketching some aspects.

8.1 Supergravity States

First consider the supergravity fluctuations. The starting point is either eleven dimensional supergravity on $AdS_3 \times S^2 \times M_6$, or IIB supergravity on $AdS_3 \times S^3 \times M_4$. For definiteness, we focus on the former; the approach in the two cases is very similar. After reduction on M_6 one has a five dimensional supergravity theory with some number of vectormultiplets n_V; hypermultiplets n_H; and gravitino multiplets n_S;[14] in addition to the gravity multiplet. The multiplicities of each multiplet are determined by the Hodge numbers of M_6, and are summarized in Table 1.

The next step is to expand in harmonics on the S^2, to get an AdS_3 spectrum of fields. The modes appearing in the expansion of each field yields a representation of the symmetry group, which includes the subgroup $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \times SU(2)_R$ corresponding to the AdS_3 isometries and the R-symmetry. For the computation of the elliptic genus we just need the spectrum of chiral primaries, which are those modes obeying $\tilde{h} = \frac{1}{2}\tilde{q}^0$, where $\tilde{L}_0 = \tilde{h}$ and $\tilde{J}^3_0 = \frac{1}{2}\tilde{q}^0$ are the $SL(2,\mathbb{R})_L$ and $SU(2)_R$

Table 1 5-dimensional supergravity spectra

M_6	n_S	n_V	n_H
CY_3	0	$h^{1,1}-1$	$2h^{1,2}+2$
$K3\times T^2$	2	22	42
T^6	6	14	14

[14] Gravitino multiplets are present for $M_6 = T^6$ or $K3 \times T^2$ to capture the extra supersymmetry.

Table 2 Spectrum of (non-singleton) chiral primaries for $AdS_3 \times S^2 \times M_6$

$s = h - \tilde{h}$	degeneracy	range of $\tilde{h} = \frac{1}{2}\tilde{q}^0$
1/2	n_H	$1/2, 3/2, \ldots$
0	n_V	$1, 2, \ldots$
1	n_V	$1, 2, \ldots$
$-1/2$	n_S	$3/2, 5/2, \ldots$
1/2	n_S	$3/2, 5/2, \ldots$
3/2	n_S	$1/2, 3/2, \ldots$
-1	1	$2, 3, \ldots$
0	1	$2, 3, \ldots$
1	1	$1, 2, \ldots$
2	1	$1, 2, \ldots$

weights. The value of $L_0 = h$ is unrestricted by the chiral primary condition, and indeed we can generate a whole tower by application of L_{-1} to a lowest h state. The details of the computation of this spectrum can be found in [91][15]; we summarize the result in Table 2. Since the chiral primaries form multiplets under $SL(2, \mathbb{R})_L$ symmetry, in Table 2 we list the spectrum of single particle chiral primaries that are also primary under the leftmoving $SL(2, \mathbb{R})$; i.e. are annihilated by L_1.

The tower of $\tilde{h}$ values correspond to the tower of spherical harmonics on S^2. This spectrum does not include the so-called singletons; we will come back to this point momentarily.

Given this spectrum it is straightforward to work out the elliptic genus as

$$\chi^{sugra} = \mathrm{Tr}_{chir.\ prim.} \left[(-1)^{\tilde{q}^0} q^{L_0} \right] \tag{196}$$

where $q = e^{2\pi i \tau}$. The sum over states includes multiparticle contributions. The result is

$$\chi^{sugra}(\tau) = M(q)^{-\mathrm{Euler}} \prod_{n=1}^{\infty} (1-q^n)^{n_v+3-2n_s} \left(1-q^{n+1}\right), \tag{197}$$

where the McMahon function is defined as

$$M(q) = \prod_{n=1}^{\infty} (1-q^n)^n, \tag{198}$$

and "Euler" denotes the Euler number of M_6.

Now we incorporate the singletons. Singleton modes are pure gauge configurations that are nonetheless physical in the presence of the AdS_3 boundary. To see why, consider the case of a $U(1)$ gauge field with Chern-Simons term. The configuration $A_w = \partial_w \Lambda(w)$ is formally pure gauge, but from (74) it carries the nonzero stress tensor $T_{ww} = \frac{k}{8\pi}(\partial_w \Lambda)^2$, and hence is physical. This is possible because the true gauge transformations must vanish at the boundary and it is only those that leave the stress

[15] The earlier references [92, 93] give incorrect ranges of $\bar{h}$ that differ slightly from these.

tensor invariant. The singleton states are described in the CFT as $J_{-1}|0\rangle$, where J is the current corresponding to A. We also have the $SL(2,\mathbb{R})$ descendants of these states.

A similar story holds for singletons associated with diffeomorphisms that are nonvanishing at the boundary. These correspond to the states $L_{-2}|0\rangle$ and $SL(2,\mathbb{R})$ descendants thereof. The explicit form of the diffeomorphisms is given in [9].

We can now work out the contribution of the singletons to the elliptic genus of the (0,4) theory. If there are n_L leftmoving currents then the contribution of singletons is

$$\chi_{NS}^{sing} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n_L}} \frac{1}{(1-q^{n+1})}. \tag{199}$$

We need to know the number of leftmoving currents, which involves knowing the form of the AdS$_3$ Chern-Simons terms. These can be worked out from reduction of the eleven dimensional theory, and gives

$$n_L = \begin{cases} 5 & T^6 \\ 21 & K3\times T^2 \\ n_V & CY_3 \end{cases} \tag{200}$$

See [14] for the derivation.

We find the full result by multiplying (197) and (199):

$$\chi_{NS} = \chi_{NS}^{sugra}\chi_{NS}^{sing} = \begin{cases} 1 & T^6 \\ 1 & K3\times T^2 \\ M(q)^{-\mathrm{Euler}}\prod_{n=1}^{\infty}(1-q^n)^3 & CY_3 \end{cases} \tag{201}$$

We find that in the T^6 and $K3\times T^2$ cases the singletons precisely cancel the dynamical contribution (197). For the CY$_3$ the dependence on n_V cancelled. Note that these conclusion are a result of cancelations between propagating states from Table 2 and the singletons.

8.2 Contribution from Wrapped Branes

The final ingredient in the computation of the elliptic genus is the contribution from wrapped branes. In the (0,4) theory corresponding to M-theory on AdS$_3\times S^2\times M_6$, these are M2-branes wrapped on 2-cycles of M_6. In [11] it was shown that this computation is equivalent to the Gopakumar-Vafa derivation [94, 95] of the topological string partition function from M-theory, and this leads to the connection between the black hole elliptic genus and the topological string. The main novelty is that both M2-branes and anti-M2-branes turn out to preserve the same supersymmetry when situated at opposite poles of the S^2. The complete contribution then takes the form of an absolute square, which in turn leads to the OSV relation between Z_{BH} and $|Z_{top}|^2$. There is much more that can be said here, but we refer the reader to [11, 87] for more details.

To bring the story to its logical conclusion, one should now try to make the explicit comparison with the (0,4) CFT, analogous to what was done in [96]. This requires an explicit result for the CFT elliptic genus, which is not available so far. We again refer the reader to the references for what is presently known.

9 Conclusion

We hope to have given the reader an understanding of how to compute the entropy of an AdS_3 black hole, and compare with CFT. One main lesson is that the success of most of the black hole/CFT entropy comparisons in the literature can be traced back to the matching of symmetries and anomalies. This gives a better understanding of *why* the entropies agree, even at the subleading level, and for certain non-supersymmetric black holes. We have also sketched the route by which one can hope to make *exact* comparisons between black hole and CFT partition functions, although much work remains to be done to bring this program to completion.

We conclude by mentioning a few open issues. In Sect. 4.3, we discussed how the entropies of fundamental heterotic strings can be deduced from a gravitational computation. The reader might be puzzled as to why we did not also consider the seemingly simpler example of type II fundamental strings. In fact, the type II case, rather than being simpler, is enigmatic. From the point of view of higher derivative terms in the spacetime action, the difference between the two cases is that R^2 corrections are absent in the type II case. But it is such R^2 terms that resolve the naked singularity of the heterotic string, replacing it by a finite size horizon. One also sees a crucial difference in our anomaly-based approach. In the type II case, spacetime rotations couple non-chirally to the string worldsheet, hence there is no anomaly inflow mechanism by which one can deduce the central charges. We are therefore unable to compute the entropy on the gravitational side. It is an important open problem as to whether higher derivative terms (e.g. R^4 terms) resolve the naked singularity of the type II string, and whether the microscopic entropy can be reproduced.

Finally, one of the main motivations for undertaking an extensive study of black entropy in string theory is to shed light on the resolution of the information paradox. The success of the AdS/CFT correspondence is usually interpreted to mean that there is no information loss, since the boundary CFT has manifestly unitary evolution, and so one can in principle track the explicit time evolution of any given microstate. A truly satisfying resolution of the information paradox will involve providing an analogous description in the bulk. In the context of the computations described here, we would like to be able to compute the AdS partition functions via an explicit sum over bulk states. The tools for such a computation are currently being developed in the context of deriving bulk states dual to CFT microstates (for reviews see [89, 97]). It will be very illuminating to see how the same AdS partition function can be computed either by summing over black hole geometries or by enumerating individual bulk states.

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97. Bena, I., Warner N.P.: *Black Holes, Black Rings and their Microstates*. Lect. Notes. Phys. **755**, x–xx (2008) 244

The Attractor Mechanism in Five Dimensions

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Abstract We give a pedagogical introduction to the attractor mechanism. We begin by developing the formalism for the simplest example of spherically symmetric black holes in five dimensions which preserve supersymmetry. We then discuss the refinements needed when spherical symmetry is relaxed. This is motivated by rotating black holes and, especially, black rings. An introduction to non-BPS attractors is included, as is a discussion of thermodynamic interpretations of the attractor mechanism.

1 Introduction

These lectures are intended as a pedagogical introduction to the attractor mechanism. With this mission in mind, we will seek to be explicit and, to the extent possible, introduce the various ingredients using rather elementary concepts. While this will come at some loss in mathematical sophistication, it should be helpful to students who are not already familiar with the attractor mechanism and, for the experts, it may serve to increase transparency.

A simple and instructive setting for studying the attractor mechanism is M-theory compactified to five dimensions on a Calabi-Yau three-fold. The resulting low energy theory has $N = 2$ supersymmetry, and it is based on real special geometry. We will focus on this setting because of the pedagogical mission of the lectures: real special geometry is a bit simpler than complex special geometry, underlying $N = 2$ theories in four dimensions.

The simplest example where the attractor mechanism applies is that of a regular, spherically symmetric black hole that preserves supersymmetry. In the first lecture, we develop the attractor mechanism in this context and then verify the results by considering the explicit black hole geometry.

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In the second lecture, we generalize the attractor mechanism to situations that preserve supersymmetry, but not necessarily spherical symmetry. Some representative examples are rotating black holes, multi-center black holes, black strings, and black rings. Each of these examples introduces new features that have qualitative significance for the implementation of the attractor mechanism. The approach will follow the paper [1] rather closely, with the difference that here we include many more examples and other pedagogical material that should be helpful when learning the subject.

In the third lecture, we consider an alternative approach to the attractor mechanism which amounts to seeing the attractor behavior as a result of an extremization procedure, rather than a supersymmetric flow. One setting that motivates this view is applications to black holes that are extremal but not supersymmetric. Extremization principles makes it clear that the attractor mechanism applies to such black holes as well.

Another reason for the interest in extremization principles is more philosophical: We would like to understand what the attractor mechanism means in terms of physical principles. There does not yet seem to be a satisfactory formulation that encompasses all the different examples, but there are many interesting hints.

The literature on the attractor mechanism is by now enormous. As general references let us mention from the outset the original works [2, 3, 4, 5] establishing the attractor mechanism. It is also worth highlighting the reviews [6, 7] which consider the subject using more mathematical sophistication than we do here. In view of the extensive literature on the subject, we will not be comprehensive when referencing. Instead, we generally provide just a few references that seem useful entry points to the literature for the student of the subject. I apologize in advance to the many authors on the subject that I fail to reference.

2 The Basics of the Attractor Mechanism

In this section, we first introduce a few concepts from the geometry of Calabi-Yau spaces and real special geometry. We then review the compactification of eleven-dimensional supergravity on a Calabi-Yau space and the resulting $N = 2$ supergravity Lagrangian in five dimensions. This sets up a discussion of the attractor mechanism for spherically symmetric black holes in five dimensions. We conclude the lecture by giving explicit formulae in the case of toroidal compactification.

2.1 Geometrical Preliminaries

On a complex manifold with hermitian metric $g_{\mu\bar{\nu}}$ it is useful to introduce the Kähler two-form J through

$$J = ig_{\mu\bar{\nu}}dz^{\mu} \wedge dz^{\bar{\nu}}. \tag{1}$$

Kähler manifolds are complex manifolds with hermitian metric such that the corresponding Kähler form is closed, $dJ = 0$. The linear space spanned by all closed (1,1) forms (modulo exact forms) is an important structure that is known as the Dolbault cohomology and given the symbol $H^{1,1}_{\bar{\partial}}$. If we denote by J_I a basis of this cohomology, we can expand the closed Kähler form as

$$J = X^I J_I; \qquad I = 1, \dots, h_{11}. \tag{2}$$

This expansion is a statement in the sense of cohomology, so it should be understood modulo exact forms.

Introducing the basis (1,1)-cycles Ω^I we can write the expression

$$X^I = \int_{\Omega^I} J \qquad ; I = 1, \dots, h_{11}, \tag{3}$$

for the real expansion coefficients X^I in (2). We see that they can be interpreted geometrically as the volumes of (1,1)-cycles within the manifold. The X^I are known as Kähler moduli. In the context of compactification the Kähler moduli become functions on spacetime, and so the X^I will be interpreted as scalar fields.

One of several ways to define a Calabi-Yau space is that it is a Kähler manifold that permits a globally defined holomorphic three-form. One consequence of this property is that Calabi-Yau spaces do not have any (0, 2) and (2, 0) forms. For this reason the (1, 1), cycles Ω^I are in fact the only two-cycles on the manifold.

The two-cycles Ω^I give rise to a dual basis of four-cycles Ω_I, $I = 1, \dots, h_{11}$, constructed such that their intersection numbers with the two-cycles are canonical $(\Omega^I, \Omega_J) = \delta^I_J$. The volumes of the four-cycles are measured by the Kähler form as

$$X_I = \frac{1}{2} \int_{\Omega_I} J \wedge J. \tag{4}$$

The integral can be evaluated by noting that the two-form J_I covers the space transverse to the 4-cycle Ω_I. Therefore

$$X_I = \frac{1}{2} \int_{CY} J \wedge J \wedge J_I = \frac{1}{2} C_{IJK} X^J X^K, \tag{5}$$

where the integrals

$$C_{IJK} = \int_{CY} J_I \wedge J_J \wedge J_K, \tag{6}$$

are known as intersection numbers because they count the points where the four-cycles Ω_I, Ω_J, and Ω_K all intersect.

2.2 The Effective Theory in Five Dimensions

We next review the compactification of M-theory on a Calabi-Yau manifold [8]. The resulting theory in five dimensions can be approximated at large distances by $N = 2$

supergravity. In addition to the $N=2$ supergravity multiplet, the low energy theory will include matter organized into a number of $N=2$ vector multiplets and hyper-multiplets. In discussions of the attractor mechanism, the hyper-multiplets decouple and can be neglected. We therefore focus on the gravity multiplet and the vector multiplets.

The $N=2$ supergravity multiplet in five dimensions contains the metric, a vector field, and a gravitino (a total of $8+8$ physical bosons+fermions). Each $N=2$ vector multiplet in five dimensions contains a vector field, a scalar field, and a gaugino (a total of $4+4$ physical bosons+fermions). It is useful to focus on the vector fields. These fields all have their origin in the three-form in eleven dimensions which can be expanded as

$$\mathscr{A} = A^I \wedge J_I \qquad ; I = 1, \ldots, h_{11}. \tag{7}$$

The J_I are the elements of the basis of (1,1) forms introduced in (2). Among the h_{11} gauge fields A^I, $I = 1, \ldots, h_{11}$, the linear combination

$$A^{\text{grav}} = X_I A^I, \tag{8}$$

is a component of the gravity multiplet. This linear combination is known as the graviphoton. The remaining $n_V = h_{11} - 1$ vector fields are components of $N=2$ vector multiplets.

The scalar components of the vector multiplets are essentially the scalar fields X^I introduced in (3). The only complication is that, since one of the vector fields does not belong to a vector multiplet, it must be that one of the scalars X^I also does not belong to a vector multiplet. Indeed, it turns out that the overall volume of the Calabi-Yau space

$$\mathscr{V} = \frac{1}{3!}\int_{CY} J \wedge J \wedge J = \frac{1}{3!} C_{IJK} X^I X^J X^K, \tag{9}$$

is in a hyper-multiplet. As we have already mentioned, hyper-multiplets decouple and we do not need to keep track of them. Therefore, (9) can be treated as a constraint that sets a particular combination of the X^Is to a constant. The truly independent scalars obtained by solving the constraint (9) are denoted $\phi^i, i = 1, \ldots, n_V$. These are the scalars that belong to vector multiplets.

We now have all the ingredients needed to present the Lagrangean of the theory. The starting point is the bosonic part of eleven-dimensional supergravity

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int \left[-R\,{}^*1 - \frac{1}{2}\mathscr{F} \wedge {}^*\mathscr{F} - \frac{1}{3!}\mathscr{F} \wedge \mathscr{F} \wedge \mathscr{A} \right], \tag{10}$$

where the four-form field strength is $\mathscr{F} = d\mathscr{A}$. The coupling constant is related to Newton's constant as $\kappa_D^2 = 8\pi G_D$. Reducing to five dimensions, we find

$$S_5 = \frac{1}{2\kappa_5^2}\int \left[-R\,{}^*1 - G_{IJ} dX^I \wedge {}^*dX^J - G_{IJ} F^I \wedge {}^*F^J - \frac{1}{3!} C_{IJK} F^I \wedge F^J \wedge A^K \right], \tag{11}$$

where $F^I = dA^I$ and $\kappa_5^2 = \kappa_{11}^2/\mathcal{V}$. The hodge-star is now the five-dimensional one, although we have not introduced new notation to stress this fact.

The gauge kinetic term in (11) is governed by the metric

$$G_{IJ} = \frac{1}{2}\int_{CY} J_I \wedge {}^* J_J. \tag{12}$$

It can be shown that

$$G_{IJ} = -\frac{1}{2}\partial_I \partial_J (\ln \mathcal{V}) = -\frac{1}{2\mathcal{V}}\left(C_{IJK}X^K - \frac{1}{\mathcal{V}}X_I X_J\right), \tag{13}$$

where the notation $\partial_I = \frac{\partial}{\partial X^I}$. Combining (5) and (9) we have the relation

$$X_I X^I = 3\mathcal{V}, \tag{14}$$

and so (13) gives

$$G_{IJ}X^J = \frac{1}{2\mathcal{V}}X_I. \tag{15}$$

The metric G_{IJ} (and its inverse G^{IJ}) thus lowers (and raises) the indices $I,J = 1,\cdots,h_{11}$. It is sometimes useful to extend this action to the intersection numbers C_{IJK} so that, e.g., the constraint (9) can be be reorganized as

$$\mathcal{V}^2 = \frac{1}{3!}C^{IJK}X_I X_J X_K, \tag{16}$$

where all indices were either raised or lowered.

The effective action in five dimensions (11) was written in terms of the fields X^I which include some redundancy because the constraint (9) should be imposed on them. An alternative form of the scalar term which employs only the unconstrained scalars ϕ^i is

$$\mathscr{L}_{\text{scalar}} = -\frac{1}{2\kappa_5^2}g_{ij}d\phi^i \wedge {}^* d\phi^j, \tag{17}$$

where the metric on moduli space is

$$g_{ij} = G_{IJ}\partial_i X^I \partial_j X^J. \tag{18}$$

Here derivatives with respect to the unconstrained fields are

$$\partial_i X^I = \frac{\partial X^I}{\partial \phi^i}. \tag{19}$$

So far we have just discussed the bosonic part of the supergravity action. We will not need the explicit form of the terms that contain fermions. However, it is important that the full Lagrangean is invariant under the supersymmetry variations

$$\delta\psi_\mu = \left[D_\mu(\omega) + \frac{i}{24}X_I\left(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\Gamma^\rho\right)F^I_{\nu\rho}\right]\epsilon, \tag{20}$$

$$\delta\lambda_i = -\frac{1}{2}G_{IJ}\partial_i X^I\left[\frac{1}{2}\Gamma^{\mu\nu}F^J_{\mu\nu} + i\Gamma^\mu\partial_\mu X^J\right]\epsilon, \tag{21}$$

of the gravitino ψ_μ and the gauginos λ_i, $i = 1,\ldots,n_V$. Here ϵ denotes the infinitesimal supersymmetry parameter. $D_\mu(\omega)$ is the covariant derivative formed from the connection ω and acting on the spinor ϵ. The usual Γ-matrices in five dimensions are denoted Γ^μ; and their multi-index versions $\Gamma^{\mu\nu}$ and $\Gamma^{\mu\nu\rho}$ are fully anti-symmetrized products of those.

2.3 A First Look at the Attractor Mechanism

We have now introduced the ingredients we need for a first look at the attractor mechanism. For now, we will consider the case of supersymmetric black holes. As the terminology indicates, such black holes preserve at least some of the supersymmetries. This means $\delta\psi_\mu = \delta\lambda_i = 0$ for some components of the supersymmetry parameter ϵ. A great deal can be learnt from these conditions by analyzing the explicit formulae (20), (21).

In order to make the conditions more explicit, we will make some simplifying assumptions. First of all, we will consider only stationary solutions in these lectures. This means we assume that the configuration allows a time-like Killing vector. The corresponding coordinate will be denoted t. All the fields are independent of this coordinate. The supersymmetry parameter ϵ satisfies

$$\Gamma^{\hat{t}}\epsilon = -i\epsilon, \tag{22}$$

where hatted coordinates refer to a local orthonormal basis. In order to keep the discussion as simple and transparent as possible, we will for now also assume radial symmetry. This last assumption is very strong and will be relaxed in the following lecture. At any rate, under these assumptions the gaugino variation (21) reads

$$\delta\lambda_i = \frac{i}{2}G_{IJ}\partial_i X^I(F^J_{m\hat{t}} - \partial_m X^J)\Gamma^m\epsilon = 0, \tag{23}$$

where m is the spatial index. We exploited that due to radial symmetry only the electric components $F^I_{m\hat{t}}$ of the field strength can be nonvanishing. We next assume that the solution preserve $N = 1$ supersymmetry so that (22) are the only projections imposed on the spinor ϵ. Then $\Gamma^m\epsilon$ will be nonvanishing for all m and the solutions to (23) must satisfy

$$G_{IJ}\partial_i X^I\left(F^J_{m\hat{t}} - \partial_m X^J\right) = 0. \tag{24}$$

This is a linear equation that depends on the bosonic fields alone. It essentially states that the gradient of the scalar field (of type J) is identified with the electric field (of the same type). This identification is at the core of the attractor mechanism. Later we will take more carefully into account the presence of the overall projection operator $G_{IJ}\partial_i X^I$ in (24). This operator takes into account that fact that no scalar field is a superpartner of the graviphoton. This restriction arises here because the index $i = 1,\dots,n_V$ enumerating the gauginos is one short of the vector field index $I = 1,\dots,n_V + 1$.

The conditions (24) give rise to an important monotonicity property that controls the attractor flow. To see this, multiply by $\partial_r\phi^i$ and sum over i. After reorganization we find

$$G_{IJ}\partial_r X^I F^J_{r\hat{t}} = G_{IJ}\partial_r X^I \partial_r X^J \geq 0. \tag{25}$$

The quantity on the right-hand side of the equation is manifestly positively definite. In order to simplify the left hand of the equation we need to analyze Gauss' law for the flux. For spherically symmetric configurations the Chern-Simons terms in the action (11) do not contribute, so the Maxwell equation is just

$$d\left(G_{IJ}{}^*F^J\right) = 0. \tag{26}$$

Using the explicit form of the metric for a radially symmetric extremal black hole in five dimensions

$$ds^2 = -f^2 dt^2 + f^{-1}\left(dr^2 + r^2 d\Omega_3^2\right), \tag{27}$$

the component form of the corresponding Gauss' law reads

$$\partial_r\left(G_{IJ} r^3 f^{-1} F^J_{r\hat{t}}\right) = 0. \tag{28}$$

This can be integrated to give the explicit solution

$$G_{IJ}F^J_{r\hat{t}} = f\cdot\frac{1}{r^3}\cdot\text{const} \equiv f\cdot\frac{Q_I}{r^3}, \tag{29}$$

for the radial dependence of the electric field. Inserting this in (25), we find the flow equation

$$\partial_r(X^I Q_I) = f^{-1} r^3 G_{IJ}\partial_r X^I \partial_r X^J \geq 0. \tag{30}$$

We can summarize this important result as the statement that the central charge,

$$Z_e \equiv X^I Q_I, \tag{31}$$

depends monotonically on the radial coordinate r. It starts as a maximum in the asymptotically flat space and decreases as the black hole is approached. This is the attractor flow.

In order to analyze the behavior of (30) close to the horizon, it is useful to write it as

$$r\partial_r Z_e = f^{-2} r^4\, \epsilon \geq 0, \tag{32}$$

where the energy density in the scalar field is

$$\epsilon = g^{rr} G_{IJ} \partial_r X^I \partial_r X^J. \tag{33}$$

According to the line element (27) an event horizon at $r = 0$ is characterized by the asymptotic behavior $f \sim r^2$. Therefore, the measure factor $f^{-2}r^4$ is finite there. Importantly, when $f \sim r^2$ the proper distance to the horizon diverges as $\int_0 dr/r$. Since the horizon area is finite, it means that the proper volume of the near horizon region diverges. This is a key property of extremal black holes. In the present discussion, the important consequence is that the energy density of the scalars in the near horizon region must vanish or else they would have infinite energy and so deform the geometry uncontrollably. We conclude that the right-hand side of (32) vanishes at the horizon, i.e. the inequality is saturated there. We therefore find the extremization condition

$$r\partial_r Z_e = 0, \text{ (at horizon)}. \tag{34}$$

This is the spacetime form of the attractor formula.

There is another form of the attractor formula that is cast entirely in terms of the moduli space. To derive it, we begin again from (24), simplify using Gauss' law (29), and introduce the central charge (31). We can write the result as

$$\partial_i Z_e = \sqrt{g_\perp} g_{ij} \partial_n \phi^j, \tag{35}$$

where $\sqrt{g_\perp} = f^{-3/2} r^3$ is the area element, g_{ij} is the metric on moduli space introduced in (18), $\partial_n = \sqrt{g^{rr}} \partial_r$ is the proper normal derivative, and the ϕ^j are the unconstrained moduli. As discussed in the previous paragraph, the energy density (33) must vanish at the horizon for extremal black holes. This means the contribution from each of the unconstrained moduli must vanish by itself, and so the right-hand side of (35) must vanish for all values of index i. We can therefore write the attractor formula as an extremization principle over moduli space

$$\partial_i Z_e = 0, \text{ (at horizon)}. \tag{36}$$

This form of the attractor formula determines the values X^I_{ext} of the scalar fields at the horizon in terms of the charges Q_I.

We can solve (36) explicitly. In order to take the constraint (9) on the scalars properly into account, it is useful to rewrite the extremization principle as

$$D_I Z_e = 0, \text{ (at horizon)}, \tag{37}$$

where the covariant derivative is defined as

$$D_I Z_e = \left(\partial_I - \frac{1}{3} (\partial_I \ln \mathscr{V}) \right) Z_e = \left(\partial_I - \frac{1}{3\mathscr{V}} X_I \right) Z_e = Q_I - \frac{1}{3\mathscr{V}} X_I Z_e. \tag{38}$$

We see that $Q_I \propto X_I$ at the attractor point, with the constant of proportionality determined by the constraint (16) on the scalar. We thus find the explicit result

$$\frac{X_I^{\text{ext}}}{\mathscr{V}^{2/3}} = \frac{Q_I}{\left(\frac{1}{3!} C^{JKL} Q_J Q_K Q_L \right)^{1/3}}, \tag{39}$$

for the attractor values of the scalar fields in terms of the charges. As a side product we found

$$\frac{Z_e^{\rm ext}}{\mathscr{V}^{1/3}} = 3\left(\frac{1}{3!}C^{JKL}Q_J Q_K Q_L\right)^{1/3}, \tag{40}$$

for the central charge at the extremum.

2.4 A Closer Look at the Attractor Mechanism

Before considering examples, we follow up on some of the important features of the attractor mechanism that we skipped in the preceding subsection: We introduce the black hole entropy, we discuss the interpretation of the central charge, and we present some details on the units.

2.4.1 Black Hole Entropy

Having determined the scalars X^I in terms of the charges we can now express the central charge (31) in terms of charges alone. It turns out that for spherically symmetric black holes the resulting expression is in fact related to the entropy through the simple formula

$$S = 2\pi \cdot \frac{\pi}{4G_5} \cdot \left(\frac{1}{3\mathscr{V}^{1/3}} Z_e^{\rm ext}\right)^{3/2}. \tag{41}$$

The simplest way to establish this relation is to inspect a few explicit black hole solutions and then take advantage of near horizon symmetries to extend the result to large orbits of black holes that are known only implicitly. The significance of the formula (41) is that it allows the determination of the black hole entropy without actually constructing the black hole geometry.

In view of the explicit expression (40) for the central charge at the extremum we find the explicit formula

$$S = 2\pi \cdot \frac{\pi}{4G_5} \cdot \sqrt{\frac{1}{3!}C^{JKL}Q_J Q_K Q_L}, \tag{42}$$

for the black hole entropy of a spherically symmetric, supersymmetric black hole in five dimensions.

2.4.2 Interpretation of the Central Charge

In the preceding subsection, we introduced the central charge (31) rather formally as the linear combination of charges that satisfies a monotonic flow. This characterization can be supplemented with a nice physical interpretation as follows. The eleven-dimensional origin of the gauge potential A_t^I can be determined from the decomposition (7). It is a three-form with one index in the temporal direction and the other two within the Calabi-Yau, directed along a (1,1)-cycle of type I. Such a three-form is sourced by $M2$-branes wrapped on the corresponding (1,1)-cycle which we have denoted Ω^I. The volume of this cycle is precisely X^I, according to (3). Putting these facts together, it is seen that the central charge (31) is the total volume of the wrapped cycles, with multiple wrappings encoded in the charge Q_I. We can interpret the underlying microscopics as a single $M2$-brane wrapping some complicated cycle Ω within the Calabi-Yau which can be characterized in terms of a decomposition

$$\Omega = Q_I \Omega^I , \tag{43}$$

on the canonical cycles Ω^I. Then the central charge is identified with the mass of this $M2$-brane, up to an overall factor of the tension.

There is yet another interpretation of the central charge which takes as starting point the $N=2$ supersymmetry algebra

$$\left\{ Q^A_\alpha, Q^B_\beta \right\} = 2\left(\delta^{AB} P_\mu \left(\Gamma^\mu\right)_{\alpha\beta} + \delta_{\alpha\beta} \epsilon^{AB} Z_e \right), \tag{44}$$

where $A, B = 1, 2$ distinguish the two supercharges. The last term on the right-hand side (proportional to Z_e) is the central term. It is introduced from a purely algebraic point of view as a term that commutes with all other generators of the algebra. The algebra is most usefully analyzed in the restframe where $P_\mu(\Gamma^\mu)_{\alpha\beta} = P_0(\Gamma^0)_{\alpha\beta}$. Consider a state that is annihilated by one or more of the supercharges Q^A_α. Taking expectation value on both sides with respect to this states, and demanding positive norm of the state, we find the famous BPS inequality

$$M = |P_0| \geq Z_e , \tag{45}$$

with the inequality saturated exactly when supersymmetry is preserved by the state. Supersymmetric black holes are BPS states, and so their mass should agree with the algebraic central charge. In the preceding paragraph, we showed that the mass agrees with the central charge introduced geometrically, so the alternate introductions of the central charge agree.

2.4.3 Some Comments on Units and Normalizations

Let us conclude this subsection with a few comments on units. It is standard to introduce the eleven-dimensional Planck length through $\kappa_{11}^2 = (2\pi)^7 l_P^9$. In this notation, the five-dimensional Newton's constant is

$$G_5 = \frac{\pi}{4} \cdot \frac{(2\pi l_P)^6}{\mathcal{V}} \cdot l_P^3 , \tag{46}$$

and the $M2$-brane tension is $\tau_{M2} = \frac{1}{(2\pi)^2 \ell_P^3}$. The relation to standard string theory units are $l_P = g_s^{1/3}\sqrt{\alpha'}$, and the radius of the M-theory circle is $R_{11} = g_s\sqrt{\alpha'}$. Now, the physical charges Q_I were introduced in (29) as the constant of integration from Gauss' law, following standard practice in supergravity. Such physical charges are proportional to quantized charges n_I according to

$$Q_I = \left(\frac{\mathcal{V}}{(2\pi l_P)^6}\right)^{-2/3} \cdot l_P^2 \cdot n_I = \left(\frac{\pi}{4G_5}\right)^{-2/3} n_I. \tag{47}$$

The mass of the brane configuration is

$$M = \tau_{M2} X^I n_I = \frac{1}{l_P^3} \cdot \frac{\mathcal{V}}{(2\pi l_P)^6} \cdot \frac{X^I}{\mathcal{V}^{1/3}} \cdot Q_I = \frac{\pi}{4G_5} \cdot \frac{X^I}{\mathcal{V}^{1/3}} \cdot Q_I. \tag{48}$$

The formulae (47), (48) are the precise versions of the informal notions that the charge Q_I counts the number of branes and that the central charge Z_e agrees with the mass. We see that there are awkward constants of proportionality, which vanish in units where $G_5 = \frac{\pi}{4}$, and the volumes of two-cycles are measured relative to $\mathcal{V}^{1/3}$. In this first lecture, we will for the most part go through the trouble of keeping all units around to make sure that it is clear where the various factors go. In later lectures, we will revert to the simplified units.[1] If needed, one can restore units by referring back to the simpler special cases.

2.5 An Explicit Example

We conclude this introductory lecture by working out a simple example explicitly. The example we consider is when the Calabi-Yau space is just a torus $CY = T^6$. Strictly speaking a torus is not actually a Calabi-Yau space, if by the latter we mean a space with exactly $SU(3)$ holonomy. The issue is that for M-theory on T^6 the effective five-dimensional theory has $N = 8$ supersymmetry rather than $N = 2$ supersymmetry as we have assumed. This means there are extra gravitino multiplets in the theory which we have not taken into account. However, these gravitino multiplets decouple from the black hole background, and so it is consistent to ignore them, in much the same way that we already ignore the $N = 2$ hypermultiplets. We can therefore use the formalism reviewed above without any change.

In the explicit example, we will further assume that the metric on the torus is diagonal so that the Kähler form takes the product form

$$J = i\left(X^1 dz^1 \wedge d\bar{z}^1 + X^2 dz^2 \wedge d\bar{z}^2 + X^1 dz^3 \wedge d\bar{z}^3\right). \tag{49}$$

Then the scalar fields X^I with $I = 1,2,3$ are just the volumes of each T^2 in the decomposition $T^6 = (T^2)^3$. The only nonvanishing intersection numbers of these two-cycles are $C_{123} = 1$ (and cyclic permutations). The constraint (9) on the scalars

[1] In fact, the supersymmetry algebra (44) was already simplified this way to avoid overly heavy notation.

therefore takes the simple form

$$X^1 X^2 X^3 = \mathscr{V}. \tag{50}$$

The volumes (5) of four-cycles on the torus are

$$X_1 = X^2 X^3 = \mathscr{V}/X^1 \quad \text{(and cyclic permutations)}. \tag{51}$$

2.5.1 Attractor Behavior

The central charge (31) for this example is

$$Z_e = X^1 Q_1 + X^2 Q_2 + X^3 Q_3. \tag{52}$$

According to the extremization principle, we can determine the scalar fields at the horizon by minimizing this expression over moduli space. The constraint (50) can be implemented by solving in terms of one of the X^Is and then extremizing (52) over the two remaining moduli. Alternatively, one can employ Lagrange multipliers. Either way, the result for the scalars in terms of the charges is

$$\frac{X_1^{\text{ext}}}{\mathscr{V}^{1/3}} = \left(\frac{Q_1^2}{Q_2 Q_3}\right)^{1/3} = \frac{Q_1}{(Q_1 Q_2 Q_3)^{1/3}} \qquad \text{(and cyclic permutations)}. \tag{53}$$

These are the horizon values for the scalars predicted by the attractor mechanism. They agree with the general formula (39). Below, we confirm these values in the explicit solutions.

At the attractor point (53), the three terms in the central charge (52) are identical. The central charge takes the value

$$Z_{\text{ext}} = 3(Q_1 Q_2 Q_3)^{1/3}. \tag{54}$$

The black hole entropy (41) becomes

$$S = 2\pi \cdot \frac{\pi}{4G_5} \cdot (Q_1 Q_2 Q_3)^{1/2} = 2\pi (n_1 n_2 n_3)^{1/2}. \tag{55}$$

This is the entropy computed using the attractor formalism, i.e. without explicit construction of the black hole geometry. At the risk of seeming heavy handed, we wrote (55) both in terms of the proper (dimensionful) charges Q_I and also in terms of the quantized charges n_I.

The entropy formula (55) is rather famous, so let us comment a little more on the relation to other work. The $M2$-brane black hole considered here can be identified, after duality to type IIB theory, with the $D1$–$D5$ black hole considered by Strominger and Vafa [9]. In this duality frame, two of the M2-brane charges become the background D-branes, and the third charge is the momentum p along the $D1$-brane. Then (55) coincides with Cardy's formula

$$S = 2\pi\sqrt{\frac{ch}{6}}, \tag{56}$$

where the central charge $c = 6N_1N_5$ for the CFT on the D-branes and $h = p$ for the energy of the excitations. In the present lectures, we are primarily interested in macroscopic features of black holes, and no further details on the microscopic theory will be needed. For more review on this, consult e.g. [10, 11, 12].

2.5.2 Explicit Construction of the Black Holes

We can compare the results from the attractor computation with an explicit construction of the black hole. The standard form of the $M2$-brane solution in eleven-dimensional supergravity is

$$ds_{11}^2 = H^{-2/3}dx_{\parallel}^2 + H^{1/3}dx_{\perp}^2. \tag{57}$$

Here the space parallel to the $M2$-brane is

$$dx_{\parallel}^2 = -dt^2 + dx_1^2 + dx_2^2, \tag{58}$$

when the spatial directions of the $M2$-brane have coordinates x_1 and x_2. The transverse space $dx_{\perp}^2$ is written similarly in terms of the remaining eight coordinates. The function H can be any harmonic on the transverse space; the specific one needed in our example is given below.

The harmonic function rule states that composite solutions can be formed by superimposing three $M2$-brane solutions of the form (57) with cyclically permuted choices of parallel space. The only caveat is that we must smear along all directions within the torus, i.e. the harmonic functions can depend only on the directions transverse to all the different branes. This procedure gives the standard intersecting $M2$-brane solution

$$ds_{11}^2 = -f^2dt^2 + f^{-1}\left(dr^2 + r^2d\Omega_3^2\right) + \left[\left(\frac{H_2H_3}{H_1^2}\right)^{1/3}\left(dx_1^2 + dx_2^2\right) + \text{cyclic}\right], \tag{59}$$

where

$$f = (H_1H_2H_3)^{-1/3}. \tag{60}$$

We introduced radial coordinates in the four spatial dimensions transverse to all the branes. The harmonic functions are

$$H_I = X_{I\infty} + \frac{Q_I}{r^2} \qquad ; I = 1,2,3. \tag{61}$$

Comparing the intersecting brane solution (59) with the torus metric (49) we determine the scalar fields as

$$\frac{X^1}{\mathscr{V}^{1/3}} = \left(\frac{H_2H_3}{H_1^2}\right)^{1/3} \qquad \text{(and cyclic permutations)}. \tag{62}$$

The only remaining matter fields from the five-dimensional point of view are the gauge fields

$$A^I = \partial_r H_I^{-1} dt \qquad ; I = 1,2,3. \tag{63}$$

The scalar fields X^I (62) depend in a non-trivial way on the radial coordinate r. One can verify that the dependence is such that $Z_e = X^I Q_I$ is a monotonic function of the radii, but we will focus on the limiting values. The constants $X_{I\infty}$ in the harmonic functions (61) were introduced in order to obtain the correct limit as $r \to \infty$

$$X^1 \to \left(\frac{X_{2\infty}X_{3\infty}}{(X_{1\infty})^2}\right)^{1/3} \mathscr{V}^{1/3} = X^1_\infty \qquad \text{(and cyclic permutations)}. \tag{64}$$

We used the constraint (50) in the asymptotic space and the relation (51) for the volumes of four-cycles. As the horizon $(r = 0)$ is approached the moduli simplify to

$$\frac{X^1}{\mathscr{V}^{1/3}} \to \frac{X^1_{\text{hor}}}{\mathscr{V}^{1/3}} = \left(\frac{Q_2Q_3}{Q_1^2}\right)^{1/3} \qquad \text{(and cyclic permutations)}. \tag{65}$$

In view of (51), this agrees with the values (53) predicted by the attractor mechanism.

We can also compute the black hole entropy directly from the geometry (59). The horizon at $r = 0$ corresponds to a three-sphere with finite radius $R = (Q_1Q_2Q_3)^{1/6}$. Since $V_{S^3} = 2\pi^2$ for a unit three-sphere this gives the black hole entropy

$$S = \frac{A}{4G_5} = \frac{1}{4G_5} \cdot 2\pi^2 \cdot R^3 = 2\pi(n_1n_2n_3)^{1/2}. \tag{66}$$

This explicit result for the black hole entropy is in agreement with (55) computed from the attractor mechanism.

3 Black Ring Attractors

In this lecture, we generalize the discussion of the attractor mechanism to a much larger class of stationary supersymmetric black solutions to the $N = 2$ theory in five dimensions introduced in Sect. 2.2. By giving up spherical symmetry and allowing for dipole charges, we can discuss multi-center black holes, rotating black holes and, especially, black rings.

3.1 General Supersymmetric Solutions

In the last few years, many families supersymmetric solutions have been classified by exploiting g-structures. The case considered in these lectures was analyzed in

[13] (with vector multiplets added in [14] and the ungauged case exhibited in the black ring papers [15, 16, 17, 18]). The result was that the most general supersymmetric metric with a time-like Killing vector takes the form

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}ds_4^2, \tag{67}$$

where

$$ds_4^2 = h_{mn}dx^m dx^n, \tag{68}$$

is the metric of a four-dimensional base space, and ω is a one-form on that base space. In the simplest examples, the base is just flat space, but generally it can be any hyper-Kähler manifold in four dimensions. The matter fields needed to support the solution are the field strengths $F^I = dA^I$ given by

$$F^I = d(fX^I(dt+\omega)) + \Theta^I, \tag{69}$$

and the scalar fields X^I satisfying the sourced harmonic equation

$${}^{(4)}\nabla^2(f^{-1}X_I) = \frac{1}{4}C_{IJK}\Theta^J \cdot \Theta^K, \tag{70}$$

on the base space. In these equations, Θ^I is a closed self-dual two-form $\Theta^I =^{*4} \Theta^I$ on the base. This two-form vanishes in the most familiar solutions, but in general it must be turned on. For example, it plays a central role for black rings. The inner product between two-forms is defined as the contraction

$$\alpha \cdot \beta = \frac{1}{2}\alpha_{mn}\beta^{mn}. \tag{71}$$

The self-dual part of the one-form ω introduced in the metric (67) is sourced by Θ^I according to

$$d\omega + {}^{*4}d\omega = -f^{-1}X_I\Theta^I. \tag{72}$$

The general solution specified by (67), (68), (69), (70), (71), (72) is a bit impenetrable at first sight, but things will become clearer as we study these equations. At this point, we just remark that the form of the solution given above has reduced the full set of Einstein's equation and matter equations to a series of equations that are linear if solved in the right order: first specify the hyper-Kähler base (68) and the self-dual two forms Θ^I on that base. Then solve (70) for $f^{-1}X_I$. Determine the conformal factor f of the metric from the constraint (9) and compute ω by solving (72). Finally the field strength is given in (69).[2]

[2] We need X^I which can be determined from (5). On a general Calabi-Yau, this is a nonlinear equation, albeit an algebraic one.

3.2 The Attractor Mechanism Revisited

We next want to generalize the discussion of the attractor mechanism from the spherical case considered in Sect. 2.3 to the more general solutions described above. Thus we consider the gaugino variations

$$\delta\lambda_i = \frac{i}{2} G_{IJ}\partial_i X^I \left[\frac{i}{2} F^J_{\mu\nu}\Gamma^{\mu\nu} - \partial_\mu X^J \Gamma^\mu\right]\epsilon, \tag{73}$$

$$= \frac{i}{2} G_{IJ}\partial_i X^I \left[F^J_{m\hat{t}}\Gamma^m + \frac{i}{2} F^J_{mn}\Gamma^{mn} - \partial_m X^J \Gamma^m\right]\epsilon. \tag{74}$$

In the second equation, we imposed the supersymmetry projection (22) on the spinor ϵ. In contrast to the spherically symmetric case (23), there are in general both electric $E^I_m \equiv F^I_{m\hat{t}}$ and magnetic $B^I_{mn} \equiv F^I_{mn}$ components of the field strength. However, as we explain below, it turns out that the magnetic field in fact does not contribute to (74). Therefore we have

$$\frac{i}{2} G_{IJ}\partial_i X^I \left[E^J_m - \partial_m X^J\right]\Gamma^m \epsilon = 0. \tag{75}$$

Since this is valid for all components of ϵ, we find

$$G_{IJ}\partial_i X^I \left[E^J_m - \partial_m X^J\right] = 0, \tag{76}$$

just like (24) for the spherical symmetric case. In particular, we see that the gradient of the scalar field is related to the electric field quite generally. Of course this can be seen already from the explicit form (69) of the field strength, which can be written in components as

$$E^I_m \equiv F^I_{m\hat{t}} = f^{-1}\partial_m (f X^I), \tag{77}$$

$$B^I_{mn} \equiv F^I_{mn} = f X^I (d\omega)_{mn} + \Theta^I_{mn}. \tag{78}$$

The point here is that we see how the relation (77) between the electric field and the gradient of scalars captures an important part of the attractor mechanism even when spherical symmetry is given up.

The key ingredient in reaching this result was the claim that the magnetic part (78) does not contribute to the supersymmetry variation (74). It is worth explaining in more detail how this comes about. The first term in (78) is of the form $F^I_{mn} \propto X^I (d\omega)_{mn}$. This term cancels from (74) because

$$G_{IJ}\partial_i X^I X^J = 0, \tag{79}$$

due to special geometry. Let us prove this. Lowering the index using the metric (15) we can use (5) to find

$$X_I \partial_i X^I = \frac{1}{2} C_{IJK} X^J X^K \partial_i X^I = \frac{1}{3!} \partial_i (C_{IJK} X^I X^J X^K) = 0. \tag{80}$$

due to the constraint (9) on the scalars X^I. This is what we wanted to show.

We still need to consider the second term in (78), the one taking the form $F^I_{mn} \propto \Theta^I_{mn}$. This term cancels from the supersymmetry variation (74) because the supersymmetry projection (22) combines with self-duality of Θ^I_{mn} to give

$$\Theta^I_{mn} \Gamma^{mn} \epsilon = 0. \tag{81}$$

In order to verify this, recall that the $SO(4,1)$ spinor representation can be constructed from the more familiar $SO(3,1)$ spinor representation by including Lorentz generators from using the chiral matrix $\Gamma^4 \equiv \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. All spinors that survive the supersymmetry projection (22) therefore satisfy $\Gamma^{1234}\epsilon = \epsilon$ by construction, and this means the Θ_{12} term in (81) cancels the Θ_{34} term, etc.

After this somewhat lengthy and technical aside, we return to analyzing the conditions (76). Following the experience from the spherically symmetric case, we would like to trade the electric field for the charges, by using Gauss' law. The Lagrangean (11) gives the Maxwell equation

$$d\left(G_{IJ}{}^* F^J\right) = \frac{1}{2} C_{IJK} F^J \wedge F^K, \tag{82}$$

with the source on the right-hand side arising from the Chern-Simons term. Considering the coefficient of the purely spatial four-form, we find Gauss' law

$$\nabla^m \left(f^{-1} E_{mI}\right) = -\frac{1}{8} C_{IJK} \Theta^J \cdot \Theta^K. \tag{83}$$

In arriving at this result, we must take into account off-diagonal terms in the metric (67) due to the shift by ω of the usual time element dt. These contributions cancel with the terms coming from the first term in the field strength (78). Effectively, this means only the term of the form $F^I_{mn} \sim \Theta^I_{mn}$ remains, and it is those terms that give rise to the inhomogenous terms in (83). The physical interpretation is that the electric field is sourced by a distributed magnetic field which we may interpret as a delocalized charge density.

We are now ready to derive the generalized flow equation. Multiplying (76) by $\partial_n \phi^i$ and contract with the base metric h^{mn}, we find

$$\partial^m X^I E_{mI} = G_{IJ} \partial^m X^I \partial_m X^J, \tag{84}$$

which can be reorganized as

$$\nabla^m (X^I f^{-1} E_{mI}) - X^I \nabla^m \left(f^{-1} E_{mI}\right) = f^{-1} G_{IJ} \partial^m X^I \partial_m X^J, \tag{85}$$

and then Gauss' law (83) gives

$$\nabla^m (X^I f^{-1} E_{mI}) = f^{-1} G_{IJ} \partial^m X^I \partial_m X^J - \frac{X^I}{8} C_{IJK} \Theta^J \cdot \Theta^K. \tag{86}$$

This is the generalized flow equation. In the case where $\Theta^I = 0$ the right-hand side is positive definite, and then the flow equation generalizes the monotonicity property found in (30) to many cases without radial symmetry. However, the most general case has nonvanishing Θ^I, and such general flows are more complicated.

3.3 Charges

In order to characterize the more general flows with precision, it is useful to be more precise about how charges are defined.

Consider some bounded spatial region V. It is natural to define the electric charge in the region by integrating the electric flux through the boundary ∂V as

$$Q_I(V) = \frac{1}{2\pi^2} \int_{\partial V} dS f^{-1} n^m E_{mI}, \tag{87}$$

where n^m is an outward pointing normal on the boundary. If we consider two nested regions $V_2 \subset V_1$, we have

$$Q_I(V_1) - Q_I(V_2) = -\frac{1}{16\pi^2} \int d^4x \sqrt{h} C_{IJK} \Theta^J \cdot \Theta^K, \tag{88}$$

where the second step used Gauss' law (83). This means the charge is monotonically decreasing as we move to larger volumes. The reason that it does not have to be constant is that in general the delocalized source on the right-hand side of (83) contributes.

The central charge is constructed from the electric charges by dressing them with the scalar fields. It was originally introduced in (31), but in analogy with the definition (87) of the electric charge in a volume of space, we may dress the electric field by the scalars as well and so introduce the central charge in a volume of space as

$$Z_e(V) = \frac{1}{2\pi^2} \int_{\partial V} dS f^{-1} n^m X^I E_{mI}. \tag{89}$$

Considering again a nested set of regions, we can use (86) to show that the central charge satisfies

$$Z_e(V_1) - Z_e(V_2) = \frac{1}{2\pi^2} \int d^4x \sqrt{h} \left[f^{-1} G_{IJ} \nabla^m X^I \nabla_m X^J - \frac{1}{8} C_{IJK} X^I \Theta^J \cdot \Theta^K \right]. \tag{90}$$

When $\Theta^I = 0$ the central charge is monotonically increasing as we move outward. This generalizes the result from the spherically symmetric case to all cases where the two-forms vanish. When the system is not spherically symmetric there is no unique "radius" but this is circumvented by the introduction of nested regions, which gives an orderly sense of moving "outwards". Note that in general, we do not force the nested volumes to preserve topology. In particular, there can be multiple singular points, and these then provide natural centers of the successive nesting.

When the two-forms $\Theta^I \neq 0$ the electric central charge (89) may not be monotonic, and the flow equation does not provide any strong constraint on the flow.

In order to interpret the Θ^Is properly, we would like to associate charges with them as well. Since they are two-forms, it is natural to integrate them over two-spheres and so define

$$q^I = -\frac{1}{2\pi}\int_{S^2} \Theta^I. \tag{91}$$

Since the two-forms Θ^I are closed, the integral is independent under deformations of the two-cycle and, in particular, it vanishes unless the S^2 is non-contractible on the base space. One way such non-trivial cycles can arise is by considering non-trivial base spaces. For our purposes, the main example will be when the base space is flat but endowed with singularities along one or more closed curves (including lines going off to infinity). This situation also gives rise to noncontractible S^2s because in four Euclidean dimensions a line can be wrapped by surfaces that are topologically a two-sphere.

The charges q^I defined in (91) can be usefully thought as a magnetic charges. In our main example of a flat base space with a closed curve, we may interpret the configuration concretely in terms of electric charge distributed along the curve. Since the curve is closed there is in general no net electric charge, but there will be a dipole charge, and it is this dipole charge that we identify as the magnetic charge (91).

In keeping with the analogy between the electric and magnetic charges, we would also like to introduce a magnetic central charge. The electric central charge (89) was obtained by dressing the ordinary charge (87) by the moduli. In analogy, we construct the magnetic central charge

$$Z_m(V) = -\frac{1}{2\pi}\int_{S^2} X_I \Theta^I. \tag{92}$$

In some examples, this magnetic central charge will play a role analogous to that played by the electric central charge in the attractor mechanism.

A general configuration can be described in terms of its singularities on the base space. There may be a number of isolated point-like singularities, to which we assign electric charges, and there may be a number of closed curves (including lines going off to infinity), to which we assign magnetic charges.

In four dimensions, electric and magnetic charges are very similar: They are related by electric magnetic duality, which is implemented by symplectic transformations in the complex special geometry. In five dimensions, the situation is more complicated because the Chern-Simons term makes the symmetry between point-like electric sources and string-like magnetic sources more subtle. Therefore, we will need to treat them independently.

3.4 Near Horizon Enhancement of Supersymmetry

There is another aspect of attractor behavior that we have not yet developed: The attractor leads to enhancement of supersymmetry [19]. This is a very strong condition that completely determines the attractor behavior, even when dipole charges are turned on.

The enhancement of supersymmetry means the *entire* supersymmetry of the theory is preserved near the horizon. To appreciate why that is such a strong conditions, recall the origin of the attractor flow: We considered the gaugino variation (74) and found the flow by demanding that the various terms cancel. The enhancement of supersymmetry at the attractor means each term vanishes by itself.

We first determine the supersymmetry constraint on the gravitino variation (20). By considering the commutator of two variations [19], it can be shown that the near horizon geometry must take the form $AdS_p \times S^q$. In five dimensions there are just two options: $AdS_3 \times S^2$ or $AdS_2 \times S^3$.

The near horizon geometry of the supersymmetric black hole in five dimension that we considered in Sect. 2.5 is indeed $AdS_2 \times S^3$ [20] (up to global identifications). A more stringent test is the attractor behavior of the supersymmetric rotating black hole. One might have expected that rotation would squeeze the sphere and make it oblate, but this would not be consistent with enhancement of supersymmetry. In fact, it turns out that, for supersymmeric black holes, the near horizon geometry indeed remains $AdS_2 \times S^3$ [21] (up to global identifications).

There are also examples of a supersymmetric configurations with near horizon geometry $AdS_3 \times S^2$. The simplest example is the black string in five dimensions. A more general solution is the supersymmetric black ring, which also has near horizon geometry $AdS_3 \times S^2$. Indeed, the extrinsic curvature of the ring becomes negligible in the very near horizon geometry, so there the black ring reduces to the black string. We will consider these examples in more detail in the next section.

The pattern that emerges from these examples is that black holes correspond to point-like singularities on the base and a near horizon geometry $AdS_2 \times S^3$ in the complete space. On the other hand, black strings and black rings correspond to singularities on a curve in the base and a near horizon geometry $AdS_3 \times S^2$ in the complete space. The two classes of examples are related by electric-magnetic duality which, in five dimensions, interchanges one-form potentials with two-form potentials and so interchanges black holes and black strings. This duality interchanges $AdS_3 \times S^2$ with $AdS_2 \times S^3$.

So far, we have just considered the constraints from the gravitino variation (20). The attractor behavior of the scalars is controlled by the gaugino variation (74) which we repeat for ease of reference

$$\delta\lambda_i = \frac{i}{2} G_{IJ} \partial_i X^I \left[F^J_{m\hat{t}} \Gamma^{m\hat{t}} + \frac{i}{2} F^J_{mn} \Gamma^{mn} - \partial_m X^J \Gamma^m \right] \epsilon. \tag{93}$$

Near horizon enhancement of supersymmetry demands that each term in this equation vanishes by itself, since no cancelations are possible when the spinor ϵ remains general. Let us consider the three conditions in turn.

The vanishing of the third term $\partial_m X^J = 0$ means X^J is a constant in the near horizon geometry. The attractor mechanism will determine the value of that constant as a function of the charges.

The first term in (93) reads

$$\partial_i X^I E_{Im} = 0, \tag{94}$$

in terms of the electric field introduced in (77). In the event that there is a point-like singularity in the base space, there is an S^3 in the near horizon geometry. Integrating the flux over this S^3 and recalling the definition (87) of the electric charge, we then find

$$\partial_i Z_e = 0, \tag{95}$$

in terms of the electric central charge (31). This is the attractor formula (36), now applicable in the near any point-like singularity in base space. We can readily determine the explicit attractor behavior as (39) near any horizon with S^3 topology.

We did not yet consider the condition that the second term in (93) vanishes. This term was considered in some detail after (78). There we found that the magnetic field $B^I_{mn} = F^I_{mn}$ has a term proportional to X^I which cancels automatically from the supersymmetry conditions, due to special geometry relations. However, there is also another term in B^I_{mn} which is proportional to Θ^I_{mn}. This term also cancels from the supersymmetry variation, but only for the components of the supersymmetry generator ϵ that satisfy the projection (22). However, in the near horizon region there is enhancement of supersymmetry and so the variation must vanish for all components ϵ. This can happen if the two-forms Θ^I take the special form

$$\Theta^I = kX^I, \tag{96}$$

where k is a constant (I-independent) two-form because then special geometry relations will again guarantee supersymmetry. The special form (96) will determine the scalars completely.

Indeed, suppose that sources are distributed along a curve in the base space. Then we can integrate (96) along the S^2 wrapping the curve. This gives

$$q^I = -\frac{1}{2\pi}\int_{S^2}\Theta^I = X^I_{\text{ext}}\cdot\text{constant}, \tag{97}$$

for the dipole charges in the near horizon region. The constant of proportionality is determined by the constraint (9) and so we reach the final result[3]

$$X^I_{\text{ext}} = \frac{q^I}{\left(\frac{1}{3!}C_{JKL}q^J q^K q^L\right)^{1/3}}, \tag{98}$$

for the scalar field in terms of the dipole charges. The result is applicable near singularities distributed along a curve in the base space. In particular, this is the attractor value for the scalars in the near horizon region of black strings and of black rings.

[3] In this Sects. 3 and 4 we use the simplified units where $\mathscr{V} = 1$ and $G_5 = \frac{\pi}{4}$. See Sect. 2.4 for details on units.

Our result (98) was determined directly in the near horizon region by exploiting the enhancement of supersymmetry there. In the case where $\Theta^I \neq 0$, we cannot understand the entire flow as a gradient flow of the electric central charge Z_e, nor are the attractor values given by extremizing Z_e. In fact, we can see that the attractor values (98) amount to extremization of the *magnetic* central charge (92). However, the significance of this is not so clear, since it is only the near horizon behavior that is controlled by Z_m, not the entire flow. It would be interesting to find a more complete description of the entire flow in the most general case. For now, we understand the complete flow when $\Theta^I = 0$, and the attractor behavior when $\Theta^I \neq 0$.

There is in fact another caveat we have not mentioned so far. Our expression (39) for the scalars at the electric attractor breaks down when $C^{JKL}Q_JQ_KQ_L = 0$, and similarly (98) for the magnetic attractor breaks down when $C_{IJK}q^Iq^Jq^K = 0$. In the electric case, the issue has been much studied: the case where $C^{JKL}Q_JQ_KQ_L = 0$ corresponds to black holes with area that vanishes classically. These are the small black holes. In some cases, it is understood how higher derivative corrections to the action modify the attractor behavior such that the geometry and the attractor values of the scalars become regular [22, 23, 24, 25]. The corresponding magnetic case $C_{IJK}q^Iq^Jq^K = 0$ corresponds to small black rings. This case has been studied less, but it is possible that a similar picture applies in that situation.

3.5 Explicit Examples

In this subsection, we consider a number of explicit geometries. In each example we first determine the attractor behavior abstractly, by applying the attractor mechanism, and then check the results by inspecting the geometry.

3.5.1 The Rotating Supersymmetric Black Hole

The simplest example of the attractor mechanism is the spherically symmetric black hole discussed in detail in Sect. 2.5. The generalization of the spherically symmetric solution to include angular momentum are the rotating supersymmetric black hole in five-dimensions. This solution is known as the BMPV black hole [26].

Let us consider the attractor mechanism first. The rotating black hole is electrically charged, but there are no magnetic charges, so the two-forms Θ^I vanish in this case. We showed in Sect. 3.2 that then the electric central charge Z_e must be monotonic just as it was in the nonrotating case. Extremizing over moduli space, we therefore return to the values (53) of the scalars found in the nonrotating case. Alternatively, we can go immediately to the general result (39) which is written for a general Calabi-Yau three-fold. Either way, we see that the attractor values of the scalars are independent of the black hole angular momentum. Since the rotation deforms the black hole geometry, this result is not at all obvious. The independence of angular momentum is a prediction of the attractor mechanism.

We can verify the result by inspecting the explicit black hole solution. The metric takes the form (67) where the base space dx_4^2 is just flat space R^4 which can be written in spherical coordinates as

$$dx_4^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2). \tag{99}$$

Although the solution is rotating, it is almost identical to the nonrotating example discussed in Sect. 2.5: The conformal factor f is given again by (60) where the harmonic functions H_I are given by (61). Additionally, the matter fields remain the scalar fields (62) and the gauge fields (63). The only effect of adding rotation is that now the one-forms ω are

$$\omega = -\frac{J}{r^2}(\cos^2\theta d\phi + \sin^2\theta d\psi). \tag{100}$$

As an aside, we note that the self-dual part of $d\omega$ vanishes, as it must for solutions with $\Theta^I = 0$, but the anti-selfdual part is non-trivial: It carries the angular momentum.

Now, for the purpose of the attractor mechanism, we are especially interested in the scalar fields. As just mentioned, these take the form (62) in terms of the harmonic functions, independently of the angular momentum. This means they will in fact approach the attractor values (53) at the horizon. In particular, the result is independent of the angular momentum, as predicted by the attractor mechanism.

3.5.2 Multi-center Black Holes

From the supergravity point of view, the $M2$-brane solution (57) is valid for *any* harmonic function H on the transverse space. Similarly, the intersecting brane solution (59) (and its generalization to an arbitrary Calabi-Yau three-fold) remains valid for more general harmonic functions H_I. In particular, the standard harmonic functions (61) can be replaced by

$$H_I = X_{I\infty} + \sum_{i=1}^{N} \frac{Q_I^{(i)}}{|\boldsymbol{r} - \boldsymbol{r}_i|^2}. \tag{101}$$

where $\boldsymbol{r}_i$ are position vectors in the transverse space. We will assume that all $Q_I^{(i)} > 0$ so that the configuration is regular.

The interpretation of these more general solutions is that they correspond to multi-center black holes, i.e. N black holes coexisting in equilibrium, with their gravitational attraction canceled by repulsion of the charges. The black hole centered at $\boldsymbol{r}_i$ has charges $\{Q_I^{(i)}\}$.

The attractor behavior of these solutions is the obvious generalization of the single center black holes. The attractor close to each center depends only on the charges associated with that center, because the charge integrals (87) are defined with respect to singularities on the base manifolds. This immediately implies that the attractor values for the scalars in a particular attractor region are (39) in terms of the charges $\{Q_I^{(i)}\}$ associated with this particular region.

The explicit solutions verify this prediction of the attractor mechanism because the harmonic functions (101) are dominated by the term corresponding to a single center in the attractor regime corresponding to that center.

In some ways, the multi-center solution is thus a rather trivial extension of the single-center solution. The reason it is nevertheless an interesting and important example is the following. Far from all the black holes, the geometry of the multi-center black hole approaches that of a single center solution with charges $\{Q_I\} = \{\sum_{i=1}^N Q_I^{(i)}\}$. Based on the asymptotic data alone, one might have expected an attractor flow governed by the corresponding central charge $Z_e = X^I Q_I$, leading to the attractor values for the scalars depending on the Q_I in a unique fashion, independently of the partition of the geometry into constituent black holes with charges $\{Q_I^{(i)}\}$. The multi-center black hole demonstrates that this expectation is false: The asymptotic behavior does *not* uniquely specify the attractor values of the scalars and nor does it define the near horizon geometry and the entropy.

More structure appears when one goes beyond the focus on attractor behavior and consider the full attractor flow of the scalars. As we discussed in Sect. 3.2, the flow of the scalars is a gradient flow controlled by the electric central charge (this is when the dipoles vanish). The central charge is interpreted as the total constituent mass. For generic values of the scalar fields the actual mass of the configuration is smaller, i.e. the black holes are genuine bound states. Now, in the course of the attractor flow, the values of the scalars change. At some intermediate point, it may be that the actual mass of the black hole is identical to that of two (or more) clusters of constituents. This is the point of marginal stability. There the attractor flow will split up, and continue as several independent flows, each controlled by the appropriate sets of smaller charges. This process then continues until the true attractor basins are reached. The total flow is referred to as the split attractor flow. It has interesting features which are beyond the scope of the present lecture. We refer the reader to the original papers [27, 28, 29] and the review [6].

3.5.3 Supersymmetric Black Strings

The black string is a five dimensional solution that takes the form

$$ds_5^2 = f^{-1}(-dt^2 + dx_4^2) + f^2(dr^2 + r^2 d\Omega_2^2), \tag{102}$$

where the conformal factor

$$f = \frac{1}{3!} C_{IJK} H^I H^J H^K, \tag{103}$$

in terms of the harmonic function

$$H^I = X_\infty^I + \frac{q^I}{2r}. \tag{104}$$

The geometry is supported by the gauge fields

$$A^I = -\frac{1}{2}q^I(1+\cos\theta)d\phi, \tag{105}$$

and the scalar fields

$$X^I = f^{-1}H^I. \tag{106}$$

The black string solution is the long distance representation of an M5-brane that wraps the four-cycle $q^I\Omega_I$ inside a Calabi-Yau threefold and has the remaining spatial direction aligned with the coordinate x^4. This configuration plays in important role in microscopic considerations of the four dimensional black hole (see e.g. [30]).

The gauge field (105) corresponds to the field strength $F^I = -q^I \sin\theta d\theta d\phi$. This is a magnetic field, with normalization of the charge in agreement with the one introduced in (91). The black string is therefore an example where the two-forms $\Theta^I \neq 0$.

We should note that the metric (102) of the supersymmetric black string differs from the form (67), assumed in the analysis in this lecture. The reason that a different form of the metric applies is that the black string has a null Killing vector, whereas (67) assumes a time-like Killing vector. Nevertheless, we can think of the null case as a limiting case of the time-like one. Concretely, if there is a closed curve on the base-space of (67), the black string is the limit where the curve is deformed such that two points are taken to infinity, and only a straight line remains (i.e. the return line is fully at infinity). This limiting procedure is how the simple black string arises from the more complicated black ring solution (see following example).

Let us now examine the attractor behavior of the black string. In Sect. 3.4, we showed that near horizon enhancement of supersymmetry demands that, at the attractor, the two forms simplify to $\Theta^I = kX^I$ where k is a constant (I independent) two-form. This condition was then showed to imply the expression (98) for the scalars as functions of the magnetic charges.

We can verify the attractor behavior by inspection of the explicit solution. Taking the limit $r \to 0$ on the scalars (106), we find

$$X^I_{\text{hor}} = \frac{q^I}{(\frac{1}{3!}C_{JKL}q^J q^K q^L)^{1/3}}. \tag{107}$$

This agrees with (98) predicted by the attractor mechanism.

3.5.4 Black Rings

As the final example, we consider the attractor behavior near the supersymmetric black ring [15, 16, 17, 18]. This is a much more involved example, which in fact was the motivation for the development of the formalism considered in this lecture.

The supersymmetric black ring is charged with respect to both electric charges Q_I and dipole charges q^I. Far from the ring the geometry is dominated by the electric charges, which have the slowest asymptotic fall-off, and the value of the charges can be determined using Gauss' law (87). The dipole charges are determined according to (91) where the by S^2 is wrapped around the ring. Since the two-forms do not vanish they dominate the near horizon geometry and the near horizon values of the scalar fields become (98), as they were for the black string.

We can verify the result from the attractor mechanism by inspecting the explicit black ring solution. The metric takes the general form (67). The conformal factor f is given by (60) in terms of functions H_I which takes the form:

$$H_I = X_{I\infty} + \frac{Q_I - \frac{1}{2}C_{IJK}q^J q^K}{\Sigma} + \frac{1}{2}C_{IJK}q^J q^K \frac{r^2}{\Sigma^2}, \tag{108}$$

where

$$\Sigma = \sqrt{(r^2 - R^2)^2 + 4R^2 r^2 \cos^2\theta}. \tag{109}$$

Although H_I plays the same role as the harmonic functions in other examples they are in fact not harmonic: They satisfy equations with sources. The expression for Σ vanishes when $r = R,\ \theta = \frac{\pi}{2}$, and arbitrary ψ. Therefore, the functions H_I diverge along a circle of radius R in the base space. This is the ring.

The full solution in five dimension remains regular, due to the conformal factor. At large distances $H_I \sim X_{I\infty} + \frac{Q_I}{r^2}$ so the black ring has the same asymptotic behavior as the spherically symmetric black hole considered in Sect. 2.5. This is because the dipole charges die off asymptotically, and so H_I differs from that of a black hole only at order $\mathscr{O}\left(\frac{1}{r^4}\right)$. However, the dipole charges dominate close to the horizon.

The scalar fields in the supersymmetric black ring solution take the form

$$X_I = \frac{H_I}{(\frac{1}{3!}C^{JKL}H_J H_K H_L)^{1/3}}. \tag{110}$$

In the near horizon region where the "harmonic" functions H_I diverge, the scalars approach

$$X^I = \frac{q^I}{(\frac{1}{3!}C_{JKL}q^J q^K q^L)^{1/3}}. \tag{111}$$

This is in agreement with the prediction (98) from the attractor mechanism.

In the preceding, we defined just enough of the black ring geometry to consider the attractor mechanism. For completeness, let us discuss also the remaining features. They are most conveniently introduced in terms of the ring coordinates

$$h_{mn}dx^m dx^n = \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right], \tag{112}$$

on the base space. Roughly speaking, the x coordinate is a polar angle $x \sim \cos\theta$ that combines with ϕ to form two-spheres in the geometry. The angle along the ring is ψ, and y can be interpreted as a radial direction with $y \to -\infty$ at the horizon. In terms of these coordinates, the two form sources are

$$\Theta^I = -\frac{1}{2} q^I (dy \wedge d\psi + dx \wedge d\phi). \tag{113}$$

Integrating the expression along the S^2s, we can verify that the normalization agrees with the definition (91) of magnetic charges.

The final element of the geometry is the one-form ω introduced in (67). Its non-vanishing components are

$$\omega_\psi = -\frac{1}{R^2}(1-x^2)\left[Q_I q^I - \frac{1}{6} C_{IJK} q^I q^J q^K (3+x+y)\right], \tag{114}$$

$$\omega_\phi = \frac{1}{2} X_{I\infty} q^I (1+y) + \omega_\psi. \tag{115}$$

In five dimensions there are two independent angular momenta which we can choose as J_ϕ and J_ψ. The one form (114), (115) gives their values as

$$J_\phi = \frac{\pi}{8G_5}\left(Q_I q^I - \frac{1}{6} C_{IJK} q^I q^J q^K\right), \tag{116}$$

$$J_\psi = \frac{\pi}{8G_5}\left(2R^2 X_{I\infty} q^I + Q_I q^I - \frac{1}{6} C_{IJK} q^I q^J q^K\right). \tag{117}$$

These expressions will play a role in the discussion of the interpretation of the attractor mechanism in the next section.

4 Extremization Principles

An alternative approach to the attractor mechanism is to analyze the Lagrangian directly, without using supersymmetry [31]. An advantage of this method is that the results apply to all extremal black holes, not just the supersymmetric ones [32]. A related issue is the understanding of the attractor mechanism in terms of the extremization of various physical quantities.

4.1 The Reduced Lagrangian

The attractor mechanism can be analyzed without appealing to supersymetry, by starting directly from the Lagrangian. In this subsection, we exhibit the details.

We will consider just the spherically symmetric case with the metric

$$ds^2 = -f^2 dt^2 + f^{-1}(dr^2 + r^2 d\Omega_3^2). \tag{118}$$

Having assumed spherical symmetry, it follows that the gauge field strengths take the form (29). The next step is to insert the *ansatz* into the Lagrangian (11). The result will be a reduced Lagrangian that depends only on the radial variable. In order to take advantage of intuition from elementary mechanics, it is useful to trade the radial coordinate for an auxiliary time coordinate defined by

$$dr = -\frac{1}{2}r^3 d\tau \qquad ; \quad \partial_r = -\frac{2}{r^3}\partial_\tau. \tag{119}$$

Introducing the convenient notation

$$f = e^{2U}, \tag{120}$$

a bit of computation gives the reduced action

$$L d\tau = \left[-6(\partial_\tau U)^2 - G_{IJ}\partial_\tau X^I \partial_\tau X^J + \frac{1}{4}e^{4U}G^{IJ}Q_I Q_J\right] d\tau, \tag{121}$$

up to overall constants.

Imposing a specific *ansatz* on a dynamical system removes numerous degrees of freedom. The corresponding equations of motion appear as constraints on the reduced system. In the present setting the main issue is that the charges specified by the *ansatz* are the momenta conjugate to the gauge fields. The correct variational principle is then obtained by a Legendre transform which, in this simple case, simply changes the sign of the potential in (121). Thus the equations of motion of the reduced system can be obtained in the usual way from the effective Lagrangean

$$\mathscr{L} = \left[-6(\partial_\tau U)^2 - G_{IJ}\partial_\tau X^I \partial_\tau X^J - \frac{1}{4}e^{4U}G^{IJ}Q_I Q_J\right]. \tag{122}$$

It is instructive to rewrite the effective potential in (121) and (122). Using the relations (14), (15), we can show the identity

$$G^{IJ}Q_I Q_J = \frac{2}{3}Z_e^2 + G^{IJ}D_I Z_e D_J Z_e, \tag{123}$$

where we used the definition (31) of the electric central charge Z_e and (38) of the covariant derivative on moduli space. The Lagrangean (122) can be written as

$$\mathscr{L} = -6(\partial_\tau U)^2 - G_{IJ}\partial_\tau X^I \partial_\tau X^J - \frac{1}{6}e^{4U}Z_e^2 - \frac{1}{4}e^{4U}G^{IJ}D_I Z_e D_J Z_e \tag{124}$$

$$= -6\left(\partial_\tau U \pm \frac{1}{6}e^{2U}Z_e\right)^2 \tag{125}$$

$$- G_{IJ}\left(\partial_\tau X^I \pm \frac{1}{2}e^{2U}G^{IK}D_K Z_e\right)\left(\partial_\tau X^J \pm \frac{1}{2}e^{2U}G^{JL}D_L Z_e\right) \pm \partial_\tau\left(e^{2U}Z_e\right),$$

where we used[4]

[4] We can verify this by writing $D_I Z_e = \mathscr{V}^{1/3}\partial_I(\mathscr{V}^{-1/3}Z_e)$. This amounts to changing into physical coordinates before taking the derivative and then changing back.

$$\partial_\tau X^I D_I Z_e = \partial_\tau Z_e. \tag{126}$$

Thus the Lagrangean can be written a sum of squares, up to a total derivative. We can therefore find extrema of the action by solving the linear equations of motion

$$\partial_\tau U = -\frac{1}{6} e^{2U} Z_e, \tag{127}$$

$$\partial_\tau X^I = -\frac{1}{2} e^{2U} G^{IJ} D_J Z_e. \tag{128}$$

The second equation is identical to the condition (35) that the gaugino variations vanish, as one can verify by identifying variables according to the various notations we have introduced. The first equation can be interpreted as the corresponding condition that the gravitino variation vanish. To summarize, we have recovered the conditions for supersymmetry by explicitly writing the bosonic Larangean as a sum of squares so that extrema can be found by solving certain linear equations of motion. The analysis of these linear equations can now be repeated from Sect. 2.3. In particular, finite energy density at the horizon (or enhancement of supersymmetry, as discussed in Sect. 3.4) implies the conditions $D_I Z_e = 0$, and these in turn lead to the explicit form (40) for the attractor values of the scalars.

One of the advantages of this approach to the attractor mechanism is that it applies even when supersymmetry is broken. To see this, consider solutions with constant value of the scalar fields throughout spacetime $\partial_\tau X^I = 0$. Extremizing the Lagrangian with respect to the scalar fields can then be found by considering just the potential (123). Upon variation we find

$$\left(\frac{2}{3} G_{IJ} Z_e + D_I D_J Z_e\right) D^J Z_e = 0. \tag{129}$$

This equation is solved automatically for $D_J Z_e = 0$. Such geometries are the supersymmetric solutions that have been our focus. However, it is seen that there can also be solutions where the scalars satisfy

$$\frac{2}{3} G_{IJ} Z_e + D_I D_J Z_e = 0. \tag{130}$$

Such solutions do not preserve supersymmetry, but they do exhibit attractor behavior.

4.2 Discussion: Physical Extremization Principles

In Sect. 3.4, we found that the attractor values are determined by extremizing one of the two central charges. For $\Theta^I = 0$ they are determined by extremizing the electric central charge (31) over moduli space $\partial_i Z_e = 0$. On the other hand, for $\Theta^I \neq 0$, we should instead extremize the magnetic central charge $\partial_i Z_m = 0$. These prescriptions are mathematically precise, but they lack a clear physical interpretation. It would be nice to reformulate the extremization principles in terms of physical quantities.

Let us consider first the situation when $\Theta^I = 0$. As discussed in Sect. 2.4, the electric central charge can be interpreted as the mass of the system. Therefore, extremization amounts to minimizing the mass. If we think about the attractor mechanism this way, the monotonic flow of the electric central charge amounts to a roll down potential, with scalars ultimately taking the value corresponding to dynamical equilibrium. In particular, if the scalars are adjusted to their attractor values already at infinity (these configurations are referred to as “double extreme black holes”), there is no flow because the configuration remains in its equilibirum.

A difficulty with this picture is the fact that the situation with $\Theta^I \neq 0$ works very differently even though the asymptotic configuration is in fact independent of the dipole charges. We would like a physical extremization principle that works for that case as well. The case where $\Theta^I \neq 0$ is elucidated by considering the combination

$$J_\psi - J_\phi = R^2 X_{I\infty} q^I = R^2 Z_m, \tag{131}$$

of the angular momenta (116), (117). This quantity can be interpreted as the intrinsic angular momentum of the black ring, not associated with the surrounding fields. The interesting point is that extremizing Z_m is the same as extremizing $J_\psi - J_\phi$ with R^2 fixed. It may at first seem worrying that we propose extremizing angular momenta. For a black hole, these would be quantum numbers measurable at infinity, and so they would be part of the input that specifies solution. However, the black ring solution is different: We can choose its independent parameters as q^I, Q_I, R^2 with the understanding that then the angular momenta J_ϕ and J_ψ that support the black ring must be those determined by (116), (117). The precise values of J_ϕ and J_ψ so determined depend on the scalars and the proposed extremization principle is that the scalars at the horizon are such that the combination (131) is minimal.

The proposed principle is quite similar to the extremization of the mass in the electric case of supersymmetric black holes. In fact, the combination (131) of angular momenta that we propose extremizing in the magnetic case behaves very much like a mass: it can be interpreted as the momentum along the effective string that appears in the near ring limit [33, 34, 35].

In order to elevate the extremization of (131) to a satisfying principle, one would need a geometric definition of the ring radius R that works independently of the explicit solution. Ideally, there should be some kind of conserved integral, akin to those defining the electric charges, or the more subtle ones appearing for dipole charges [36]. Another issue is that of more complicated multiple ring solutions, which are characterized by several radii. This latter problem is completely analogous to the ambiguity with assigning mass for multi-black hole solutions: the asymptotics does not uniquely specify the near horizon behavior. We will put these issues aside for now and seek an extremization principle that combines the extremization of (131) in the magnetic case with extremization of the mass in electric case and works in any basin of attraction, whether electric or magnetic in character.

To find such a principle, recall that the black hole entropy (41) can be written in terms of the central charge in the electric case. Accordingly, the extremization over moduli space can be recast as[5]

$$\partial_i S = 0. \tag{132}$$

The black ring entropy can be written compactly as

$$S = 2\pi\sqrt{J_4}. \tag{133}$$

For toroidal compactification[6] J_4 is the quartic $E_{7(7)}$ invariant, evaluated at arguments that depend on the black ring parameters according to the identifications

$$J_4 = J_4(Q_I, q^I, J_\psi - J_\phi). \tag{134}$$

The black ring is thus related to black holes in four dimensions [33, 34].

In the present context, the point is that the extremization principle (132) applies to both electric and magnetic attractors. This provides a thermodynamic interpretation of the attractor mechanism. One obstacle to a complete symmetry between the electric and magnetic cases is that near a magnetic attractor point one must apply (132) with Q_I, q^I, and R fixed, while near an electric attractor it is Q_I and J that should be kept fixed. In either case, these are the parameters that define the solution.

There is one surprising feature of the proposed physical extremization principle: The entropy is *minimized* at the attractor point. This may be the correct physics: As one moves closer to the horizon, the geometry is closer to the microscopic data. It is also in harmony with the result that, at least in some cases, extremization over the larger moduli space that includes multi-center configurations gives split attractor flows that correspond to independent regions that have even less entropy [27, 28, 29], with the end of the flow plausibly corresponding to "atoms" that have no entropy at all [38, 39, 40].

We end with a summary of this subsection: We have proposed an extremization principle (132) that applies to both the electric (black hole) and magnetic (black ring) cases. A physical interpretation in terms of thermodynamics looks promising at the present stage of development. In order to fully establish the proposed principle one would need a more detailed understanding of general flows, including those that have magnetic charges, and one would also need a more general definition of charges.

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[5] Although (41) was given in the spherically symmetric case, it can be generalized to include angular momentum [37] (just subtract J^2 under the square root. The argument given below carries through.

[6] This statement has an obvious alternate version that applies to general Calabi-Yau spaces.

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Lectures on Black Holes, Topological Strings, and Quantum Attractors (2.0)

B. Pioline

Abstract In these lecture notes, we review some recent developments on the relation between the macroscopic entropy of four-dimensional BPS black holes and the microscopic counting of states beyond the thermodynamical, large charge limit. After a brief overview of charged black holes in supergravity and string theory, we give an extensive introduction to special and very special geometry, attractor flows and topological string theory, including holomorphic anomalies. We then expose the Ooguri-Strominger-Vafa (OSV) conjecture which relates microscopic degeneracies to the topological string amplitude, and review precision tests of this formula on "small" black holes. Finally, motivated by a holographic interpretation of the OSV conjecture, we give a systematic approach to the radial quantization of BPS black holes (i.e. quantum attractors). This suggests the existence of a one-parameter generalization of the topological string amplitude and provides a general framework for constructing automorphic partition functions for black hole degeneracies in theories with sufficient degree of symmetry.

1 Introduction

Once upon a time regarded as unphysical solutions of General Relativity, black holes now occupy the central stage. In astrophysics, there is mounting evidence of stellar size and supermassive black holes in binary systems and in galactic centers (see e.g. [1]). In theoretical particle physics, black holes are believed to dominate the high energy behavior of quantum gravity (e.g. [2]). Moreover, the Bekenstein-Hawking entropy of black holes is one of the very few clues in our hands about

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the nature of quantum gravity: Just as the macroscopic thermodynamical properties of perfect gases hinted at their microscopic atomistic structure, the classical thermodynamical properties of black holes suggest the existence of quantized micro-states, whose dynamics should account for the macroscopic production of entropy.

One of the great successes of string theory is to have made this idea precise, at least for a certain class of black holes which admittedly are rather remote from reality: supersymmetric, charged black holes can indeed be viewed as bound states of D-branes and other extended objects, whose microscopic "open-string" fluctuations account for the macroscopic Bekenstein-Hawking entropy [3]. In a more modern language, the macroscopic gravitational dynamics is holographically encoded in microscopic gauge theoretical degrees of freedom living at the conformal boundary of the near-horizon region. Irrespective of the language used, the agreement is quantitatively exact in the "thermodynamical" limit of large charge, where the counting of the degrees of freedom requires only a gross understanding of their dynamics.

While the prospects of carrying this quantitative agreement over to more realistic black holes remain distant, it is interesting to investigate whether the already remarkable agreement found for supersymmetric extremal black holes can be pushed beyond the thermodynamical limit. Indeed, this regime in principle allows to probe quantum-gravity corrections to the low energy Einstein-Maxwell Lagrangian, while testing our description of the microscopic degrees of freedom in greater detail.

The aim of these lectures is to describe some recent developments in this direction, in the context of BPS black holes in $\mathscr{N} \geq 2$ supergravity.

In Sect. 2, we give an overview of extremal Reissner-Nordström black holes, recall their embedding in string theory and the subsequent microscopic derivation of their entropy at leading order, and briefly discuss an early proposal to relate the exact microscopic degeneracies to Fourier coefficients of a certain modular form.

In Sect. 3, we recall the essentials of special geometry and describe the "attractor flow", which governs the radial evolution of the scalar fields and determines the horizon geometry in terms of asymptotic charges. We illustrate these results in the context of "very special supergravities", an interesting class of toy models whose symmetries properties allow to get very explicit results.

In Sect. 4, we give a self-contained introduction to topological string theory, which allows to compute an infinite set of higher-derivative "F-term" corrections in the low energy Lagrangian. We emphasize the wave function interpretation of the holomorphic anomaly, which underlies much of the subsequent developments.

In Sect. 5, we discuss the effects of these "F-term" corrections on the macroscopic entropy and formulate the Ooguri-Strominger-Vafa (OSV) conjecture [4], which relates these macroscopic corrections to the micro-canonical counting.

In Sect. 6, based on [5, 6], we submit this conjecture to a precision test, in the context of "small black holes": These are dual to perturbative heterotic states and can therefore be counted exactly using standard conformal field theory techniques.

Finally, in Sect. 7, motivated by a holographic interpretation of the OSV conjecture put forward by Ooguri, Vafa, and Verlinde [7], we turn to the subject of "quantum attractor flows". We give a systematic treatment of the radial quantization of

BPS black holes, and compute the exact radial wave function for a black hole with fixed electric and magnetic charges. In the course of this discussion, we find evidence for a one-parameter generalization of the usual topological string amplitude and provide a framework for constructing automorphic partition functions for black hole degeneracies in theories with a sufficient degree of symmetry, in the spirit (but not the letter) of the genus-2 modular forms discussed in Sect. 2.5. This section is based on [8, 9, 10, 11, 12, 13].

We have included a number of exercices, most of which are quite easy, which are intended to illustrate, complement, or extend the discussion in the main text. The dedicated student might learn more from solving the exercices than from pondering over the text.

2 Extremal Black Holes in String Theory

In this section, we give a general overview of extremal black holes in Einstein-Maxwell theory, comment on their embedding in string theory, and outline their microscopic description as bound states of D-branes. We also review an early conjecture that relates the exact microscopic degeneracies of BPS black holes to Fourier coefficients of a certain modular form. We occasionally make use of notions that will be explained in later sections. For a general introduction to black hole thermodynamics, the reader may consult e.g. [14, 15].

2.1 Black Hole Thermodynamics

Our starting point is the Einstein-Maxwell Lagrangian for gravity and a massless Abelian gauge field in $3+1$ dimensions,

$$S=\int d^4x\,\frac{1}{16\pi G}\left[\sqrt{-g}\,R-\frac{1}{4}F\wedge\star F\right] \tag{1}$$

Assuming staticity and spherical symmetry, the only solution with electric and magnetic charges q and p is the Reissner-Nordström black hole

$$ds^2=-f(\rho)\,dt^2+f^{-1}(\rho)\,d\rho^2+\rho^2\,d\Omega^2\ ,\quad F=p\,\sin\theta d\theta\wedge d\phi+q\,\frac{dt\wedge d\rho}{\rho^2} \tag{2}$$

where $d\Omega^2=d\theta^2+\sin^2\theta\ d\phi^2$ is the metric on the two-sphere, and $f(\rho)$ is given in terms of the ADM mass M and the charges (p,q) by

$$f(\rho)=1-\frac{2\,GM}{\rho}+\frac{p^2+q^2}{\rho^2} \tag{3}$$

For most of what follows, we set the Newton constant $G = 1$. The Schwarzschild black hole is recovered in the neutral case $p = q = 0$.

The solution (2) has a curvature singularity at $r = 0$, with diverging curvature invariant $R_{\mu\nu}R^{\mu\nu} \sim 4(p^2+q^2)^2/\rho^8$. When $M^2 < p^2+q^2$, this is a naked singularity and the solution must be deemed unphysical. When $M^2 > p^2+q^2$ however, there are two horizons at the zeros of $f(\rho)$,

$$\rho_\pm = M \pm \sqrt{M^2-p^2-q^2} \tag{4}$$

which prevent the singularity to have any physical consequences for an observer at infinity; see the Penrose diagram on Fig. 1. We shall denote by I, II, III the regions outside the horizon, between the two horizons, and inside the inner horizon, respectively. Since the time-like component of the metric changes sign twice between regions I and III, the singularity at $\rho = 0$ is time-like and may be imputed to the existence of a physical source at $\rho = 0$. This is unlike the Schwarzschild black hole, whose space-like singularity at $\rho = 0$ raises more serious concerns.

Near the outer horizon, one may approximate

$$f(\rho) = \frac{(\rho-\rho_+)(\rho-\rho_-)}{\rho^2} \sim \frac{(\rho_+-\rho_-)}{\rho_+^2} r \tag{5}$$

where $\rho = \rho_+ + r$, and the line element (2) by

$$ds^2 \sim \left[-\frac{(\rho_+-\rho_-)}{\rho_+^2} r\, dt^2 + \frac{\rho_+^2}{(\rho_+-\rho_-)} \frac{dr^2}{r} \right] + \rho_+^2\, d\Omega_2^2 \tag{6}$$

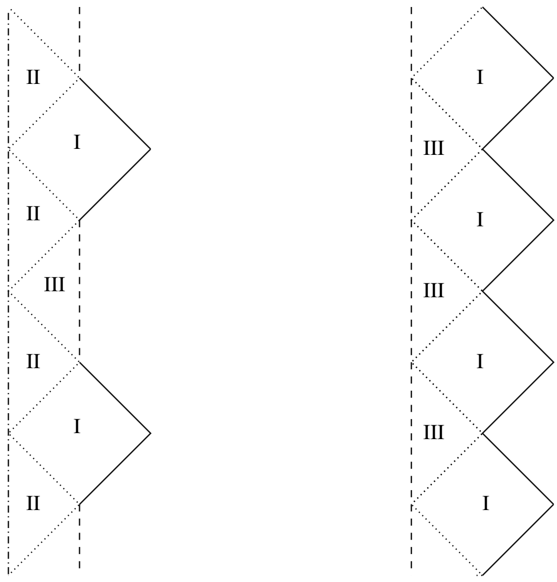

Fig. 1 Penrose diagram of the non-extremal (*left*) and extremal (*right*) Reissner-Nordström black holes. Dotted lines denote event horizons, dashed lines represent time-like singularities. The diagram on the left should be doubled along the dashed-dotted line

Defining $t = 2\rho_+^2 \tau/(\rho_+ - \rho_-)$ and $r = \eta^2$, the first term is recognized as Rindler space while the second term is a two-sphere of fixed radius,

$$ds^2 = \frac{4\rho_+^2}{\rho_+ - \rho_-} \left(-\eta^2 d\tau^2 + d\eta^2\right) + \rho_+^2 \, d\Omega_2^2. \tag{7}$$

Rindler space describes the patch of Minkowski space accessible to an observer $\mathscr{O}$ with constant acceleration κ. As spontaneous pair production takes place in the vacuum, $\mathscr{O}$ may observe only one member of that pair, while its correlated partner falls outside $\mathscr{O}$'s horizon. Hawking and Unruh have shown that, as a result, $\mathscr{O}$ detects a thermal spectrum of particles at temperature $T = \kappa/(2\pi)$, where κ is the acceleration or "surface gravity" at the horizon [16, 17]. Equivalently, smoothness of the Wick-rotated geometry $\tau \to i\tau$ requires that τ be identified modulo $2\pi i$. In terms of the inertial time t at infinity, this requires $t \sim t + i\beta$ where β is the inverse temperature

$$\beta = \frac{1}{T} = \frac{4\pi\rho_+^2}{\rho_+ - \rho_-} \tag{8}$$

Given an energy M and a temperature T, it is natural to define the "Bekenstein-Hawking" entropy S_{BH} such that $dS_{BH}/dM = 1/T$ at fixed charges.

Exercise 1. *By integrating (8), show that the entropy of a Reissner-Nordström black hole is equal to*

$$S_{BH} = \pi \left(M + \sqrt{M^2 - p^2 - q^2}\right)^2 = \pi\rho_+^2 \tag{9}$$

Remarkably, the result is, up to a factor $1/(4G)$, just equal to the area of the horizon:

$$S_{BH} = \frac{A}{4G} \tag{10}$$

This is a manifestation the following general statements, known as the "laws of black hole thermodynamics" (see e.g. [15, 18] and references therein):

(0) The temperature $T = \kappa/(2\pi)$ is uniform on the horizon;
(I) Under quasi-static changes, $dM = (T/4G)dA + \phi dq + \chi dp$;
(II) The horizon area always increases with time.

These statements rely purely on an analysis of the classical solutions to the action (1) and their singularities. The modifications needed to preserve the validity of these laws in the presence of corrections to the action (1) will be discussed in Sect. 6.2.

The analogy of (0), (I), (II) with the usual laws of thermodynamics strongly suggests that it should be possible to identify the Bekenstein-Hawking entropy with the logarithm of the number of micro-states which lead to the same macroscopic black hole,

$$S_{BH} = \log \Omega(M, p, q) \tag{11}$$

where we set the Boltzmann constant to 1. In writing this equation, we took advantage of the "no hair" theorem which asserts that the black hole geometry, after transients, is completely specified by the charges measured at infinity.

Making sense of (11) microscopically requires quantizing gravity, which for us means using string theory. As yet, progress on this issue has mostly been restricted to the case of extremal (or near-extremal) black holes, to which we turn now.

2.2 Extremal Reissner-Nordström Black Holes

In the discussion below (3), we left out one special case, namely $M^2 = p^2 + q^2$. When this happens, the inner and outer horizons coalesce in a single degenerate horizon at $r = \sqrt{p^2+q^2}$, where the scale factor vanishes quadratically:

$$f(\rho) = \left(1 - \frac{\sqrt{p^2+q^2}}{\rho}\right)^2 \sim \frac{r^2}{p^2+q^2} \tag{12}$$

Such black holes are called "extremal", for reasons that will become clear below. In this case, defining $r = (p^2+q^2)/z$, we can rewrite the near-horizon geometry as

$$ds \sim (p^2+q^2)\,\left[\frac{-dt^2+dz^2}{z^2} + d\Omega^2\right] \tag{13}$$

which is now recognized as the product of two-dimensional Anti-de Sitter space AdS_2 times a two sphere. In contrast to (6), this is now a bona-fide solution of (1). The appearance of the AdS_2 factor raises the hope that such "extremal" black holes have an holographic description, although holography in AdS_2 is far less understood than in higher dimensions (see [19] for an early discussion).

An important consequence of $f(r)$ vanishing quadratically is that the Hawking temperature (8) is zero, so that the black hole no longer radiates: This is as it should, since otherwise its mass would go below the bound

$$M^2 \geq p^2 + q^2, \tag{14}$$

producing a naked singularity. Black holes saturating this bound can be viewed as the stable endpoint of Hawking evaporation[1], assuming that all charged particles are massive. Moreover, the Bekenstein entropy remains finite

$$S_{BH} = \pi(p^2+q^2) \tag{15}$$

and becomes large in the limit of large charge. This is not unlike the large degeneracy of the lowest Landau level in condensed matter physics.

[1] The evaporation end-point of neutral black holes is far less understood and, in particular, leads to the celebrated "information paradox".

2.3 Embedding in String Theory

String theory compactified to four dimensions typically involves many more fields than those appearing in the Einstein-Maxwell Lagrangian (1). Restricting to compactifications which preserve $\mathscr{N} \geq 2$ supersymmetry in four dimensions, there are typically many Abelian gauge fields and massless scalars (or "moduli"), together with their fermionic partners, and the gauge couplings in general have a complicated dependence on the scalar fields. As a result, the static, spherically symmetric solutions are much more complicated, involving in particular a non-trivial radial dependence of the scalar fields. The first smooth solutions were constructed in the context of the heterotic string compactified on T^6 in [20], and the general solution was obtained in [21] using spectrum-generating symmetries. Charged solutions exhibit the same causal structure as the Reissner-Nordström black hole and become extremal when a certain "BPS" bound, analogous to (14), is saturated.

In fact, in the context of supergravity with $\mathscr{N} \geq 2$ extended supersymmetry, the BPS bound is a consequence of unitarity in a sector with non-vanishing central charge $Z = \sqrt{p^2+q^2}$, see (43) below. The saturation of the bound implies that the black hole preserves some fraction of the supersymmetry of the vacuum. Since the corresponding representation of the supersymmetry algebra has a smaller dimension that the generic one, such states are absolutely stable (unless they can pair up with an other extremal state with the same energy) [22]. They can be followed as the coupling is varied, which is part of the reason for their successful description in string theory.

Another peculiarity of extremal black holes in supergravity is that the radial profile of the scalars simplifies: Specifically, the values of the scalar fields at the horizon become independent of the values at infinity and depend only on the electric and magnetic charges. Moreover, the horizon area itself becomes a function of the charges only[2]. This is a consequence of the "attractor mechanism", which we will discuss at length in Sects. 3 and 7. This fits in nicely with the fact that the number of quantum states of a system is expected to be invariant under adiabatic perturbations [23]. More practically, it implies that a rough combinatorial, weak coupling counting of the micro-states may be sufficient to reproduce the macroscopic entropy.

As a side comment, it should be pointed out that even in supersymmetric theories, extremal black holes can exist which break all supersymmetries. In this case, the electromagnetic charges differ from the central charge, and the extremality bound is subject to quantum corrections. In this case, there may exist non-perturbative decay processes whereby an extremal black hole may break into smaller ones. The subject of non-supersymmetric extremal black holes has become of much interest recently, see e.g. [24, 25, 26, 27, 28, 29, 30].

[2] Although it no longer takes the simple quadratic form (15), at tree-level it is still an homogeneous function of degree 2 in the charges.

Exercise 2. *Show that if black hole of mass and charge (M,Q) breaks up into two black holes of mass and charge (M_1,Q_1) and (M_2,Q_2), then at least one of M_1/Q_1 and M_2/Q_2 must be smaller than M/Q. Conclude that quantum corrections should decrease the ratio M/Q [29, 31].*

2.4 Black Hole Counting via D-branes

The ability of string theory to account microscopically for the Bekenstein-Hawking entropy of BPS black holes (15) is one of its most concrete successes. Since this subject is well covered in many reviews, we will only outline the argument, referring e.g. to [32, 33, 34, 35] for more details and references.

The main strategy, pioneered by Strominger and Vafa [3], is to represent the black hole as a bound state of solitons in string theory, and vary the coupling so that the degrees of freedom of these solitons become weakly coupled. The BPS property ensures that the number of micro-states will be conserved under this operation.

Consider for exemple 1/8 BPS black holes in Type II string theory on T^6, or 1/4 BPS black holes on $K3\times T^2$ [36]. Both cases can be treated simultaneously by writing the compact 6-manifold as $X=Y\times S_1\times S_1'$, where $Y=T^4$ or $K3$. Now consider a configuration of Q_6 D6-branes wrapped on X, Q_2 D2-branes wrapped on $S_1\times S_1'$, Q_5 NS5-branes wrapped on $Y\times S_1$, carrying N units of momentum along S_1. The resulting configuration is localized in the four non-compact directions and supersymmetric, hence should be represented as a BPS black hole in $\mathcal{N}=8$ or $\mathcal{N}=4$ supergravity[3]. Its macroscopic entropy can be computed by studying the flow of the moduli with the above choice of charges, leading in either case to ((99) below)

$$S_{BH}=2\pi\sqrt{Q_2\,Q_5\,Q_6\,N} \tag{16}$$

The micro-states correspond to open strings attached to the D2 and D6 branes, in the background of the NS5-branes. In the limit where $Y\times S_1'$ is very small, they may be described by a two-dimensional field theory extending along the time and S_1 direction. In the absence of the NS5-branes, the open strings are described at low energy by $U(Q_2)\times U(Q_6)$ gauge bosons together with bi-fundamental matter, which is known to flow to a CFT with central charge $c=6Q_2Q_6$ in the infrared (see [34] for a detailed analysis of this point). In the presence of the NS5-branes, localized at Q_5 points along S_1', the D2-branes generally break at the points where they intersect the NS5-branes. This effectively leads to Q_5Q_2 independent D2-branes, hence a CFT with central charge $c_{\rm eff}=6Q_2Q_5Q_6$. The extremal micro-states correspond to the right-moving ground states of that field theory, with N units of left-moving momentum along S_1. By the Ramanujan-Hardy formula ((201) below), also known as the Cardy formula in the physics literature, the number of states carrying N units of momentum grows exponentially as

[3] As usual in AdS/CFT correspondence, the closed string description is valid at large value of the t'Hooft coupling g_sQ, where Q is any of the D-brane charges.

$$\Omega(Q_2,Q_5,Q_6,N) \sim \exp\left[2\pi\sqrt{\frac{c_{\rm eff}}{6}N}\right] \sim \exp\left[2\pi\sqrt{Q_2\,Q_5\,Q_6\,N}\right] \tag{17}$$

in precise agreement with the macroscopic answer (16).

While quantitatively successful, this argument has some obvious shortcomings. The degrees of freedom of the NS5-branes have been totally neglected, and the D2-branes stretching between each of the NS5-branes were treated independently. A somewhat more tractable configuration can be obtained by T-dualizing along S_1', leading to a bound state of D1-D5 branes in the gravitational background of Kaluza-Klein monopoles [37]. The latter are purely gravitational solutions with orbifold singularities, so in principle can be treated by worldsheet techniques.

Key to this reasoning was the ability to lift the 4-dimensional black hole to a 5-dimensional black string, whose ground-state dynamics can be described by a two-dimensional "black string CFT", such that Cardy's formula is applicable. This indicates how to generalize the above argument to 1/2-BPS black holes in $\mathscr{N}=2$ supergravity: any configuration of D0,D4 branes with vanishing D6-brane charge in type IIA string theory compactified on a Calabi-Yau threefold X can be lifted in M-theory to a single M5-brane wrapped around a general divisor (i.e. complex codimension one submanifold) P, with N (the D0-brane charge) units of momentum along the M-theory direction [38]. The reduction of the (0,2) tensor multiplet on the M5-brane worldvolume along the divisor P leads to a (0,4) SCFT in $1+1$ dimensions, whose left-moving central charge can be computed with some technical assumptions on P:

$$c_L = 6C(P) + c_2\cdot P \tag{18}$$

Here, $C(P)$ is the self-intersection of P, while c_2 is the second Chern class of X. Using again Cardy's formula, this leads to

$$\Omega(P,N) \sim \exp\left[2\pi\sqrt{N\left(C(P)+\frac{1}{6}c_2\cdot P\right)}\right] \tag{19}$$

To leading order, this reproduces the macroscopic computation in $\mathscr{N}=2$ supergravity, T-dual to (17),

$$S_{BH} = 2\pi\sqrt{Q_0\,C(Q_4)} \tag{20}$$

We shall return to formula (19) in Sect. 6 (Exercise 17), and show that the subleading contribution proportional to c_2 agrees with the macroscopic computation, provided one incorporates higher-derivative R^2 corrections.

2.5 Counting $\mathscr{N}$ = 4 Dyons via Automorphic Forms

While the agreement between the macroscopic entropy and microscopic counting at leading order is already quite spectacular, it is interesting to try and understand the corrections to the large charge limit. Ideally, one would like to be able to compute the exact microscopic degeneracies for arbitrary values of the charges. Here, we shall recall an interesting conjecture, due to Verlinde, Verlinde and Dijkgraaf (DVV), which purportedly relates the exact degeneracies of 1/4-BPS states in $\mathcal{N}=4$ string theory, to Fourier coefficients of a certain automorphic form [39]. This conjecture has been the subject of much recent work, which we will not be able to pay justice to in this review. However, it plays an important inspirational role for some other conjectures relating black hole degeneracies and automorphic forms, which we will develop in Sect. 7.

Consider the heterotic string compactified on T^6, or equivalently the type II string on $K3\times T^2$. The moduli space factorizes into

$$\frac{Sl(2,\mathbb{R})}{U(1)}\times\frac{SO(6,n_v,\mathbb{R})}{SO(6)\times SO(n_v)} \tag{21}$$

with $n_v=22$. The first factor is the complex scalar in the $\mathcal{N}=4$ gravitational multiplet, and corresponds to the heterotic axio-dilaton S, or equivalently to the complexified Kähler modulus of T^2 on the type II side. Points in (21) related by an action of the duality group $\Gamma=Sl(2,\mathbb{Z})\times SO(6,22,\mathbb{Z})$ are conjectured to be equivalent under non-perturbative dualities.

The Bekenstein-Hawking entropy for 1/4-BPS black holes is given by [40]

$$S_{BH}=\pi\sqrt{(\vec{q}_e\cdot\vec{q}_e)(\vec{q}_m\cdot\vec{q}_m)-(\vec{q}_e\cdot\vec{q}_m)^2} \tag{22}$$

where $\vec{q}_e$ and $\vec{q}_m$ are the electric and magnetic charges in the natural heterotic polarization. $(\vec{q}_m,\vec{q}_e)$ transform as a doublet of $SO(6,n_v)$ vectors under $Sl(2)$. Equation (22) is manifestly invariant under the continuous group $Sl(2,\mathbb{R})\times SO(6,22,\mathbb{R})$, a fortiori under its discrete subgroup Γ.

DVV proposed that the exact degeneracies should be given by the Fourier coefficients of the inverse of Φ_{10}, the unique cusp form of $Sp(4,\mathbb{Z})$ with modular weight 10:

$$\Omega\left(\vec{q}_e,\vec{q}_m\right)\overset{?}{=}\int_\gamma d\tau\,\frac{1}{\Phi_{10}(\tau)}e^{-i\left(\rho\vec{q}_m^2+\sigma\vec{q}_e^2+2\nu\vec{q}_e\cdot\vec{q}_m\right)} \tag{23}$$

Here, $\tau=\begin{pmatrix}\rho & \nu\\ \nu & \sigma\end{pmatrix}$ parameterizes Siegel's upper half plane $Sp(4,\mathbb{R})/U(2)$ and γ is the contour $0\le\rho,\sigma\le 2\pi, 0\le\nu\le\pi$. One may think of τ as the period matrix of an auxiliary genus 2 Riemann surface, with modular group $Sp(4,\mathbb{Z})$. The cusp form Φ_{10} has an infinite product representation

$$\Phi_{10}(\tau)=e^{i(\rho+\sigma+\nu)}\prod_{(k,l,m)>0}\left(1-e^{i(k\rho+l\sigma+m)}\right)^{c(4kl-m^2)} \tag{24}$$

where $c(k)$ are the Fourier coefficients of the elliptic genus of $K3$,

$$
\begin{aligned}
\chi_{K3}(\rho,\nu) &= \sum_{h\geq 0, m\in\mathbb{Z}} c(4h-m^2)e^{2\pi i(h\rho+mz)} \\
&= 24\left(\frac{\theta_3(\rho,z)}{\theta_3(\rho)}\right)^2 - 2\frac{\left(\theta_4^4(\rho)-\theta_2^4(\rho)\right)\theta_1^2(\rho,z)}{\eta^6(\rho)}\,. \qquad (25)
\end{aligned}
$$

This shows that the Fourier coefficients obtained in this fashion are (in general non-positive) integers.

The right-hand side of (23) is manifestly invariant under continuous rotations in $SO(6,22,\mathbb{R})$, hence under its discrete subgroup $SO(6,22,\mathbb{Z})$. The invariance under $Sl(2,\mathbb{Z})$ is more subtle and uses the embedding of $Sl(2,\mathbb{Z})$ inside $Sp(4,\mathbb{Z})$; using the modular invariance of Φ_{10},

$$
\Phi_{10}\left[(A\tau+B)(C\tau+D)^{-1}\right] = \left[\det(C\tau+D)\right]^{10}\ \Phi_{10}(\tau)\,, \qquad (26)
$$

one can cancel the action of $Sl(2,\mathbb{Z})$ by a change of contour $\gamma\to\gamma'$, and deform γ' back to γ while avoiding singularities.

As a consistency check on this conjecture, one can extract the large charge behavior of $\Omega(\vec{q}_e,\vec{q}_m)$ by computing the contour integral in (23) by residues and obtain agreement with (22) [39].

Exercise 3. *By picking the residue at the divisor* $D=\rho\sigma+\nu-\nu^2\sim 0$ *and using* $\Phi_{10}\sim D^2\,\eta^{24}(\rho')\eta^{24}(\sigma')/\det^{12}(\tau)$ *where* $\rho'=-\frac{\sigma}{\rho\sigma-\nu^2}$, $\sigma'=-\frac{\rho}{\rho\sigma-\nu^2}$, *reproduce the leading charge behavior (22). You may seek help from [39, 41].*

A recent "proof" of the DVV conjecture has recently been given by lifting 4D black holes with unit D6-brane charge to 5D and using the Strominger-Vafa relation between degeneracies of 5D black hole and the elliptic genus of the Hilbert scheme (or symmetric orbifold) $\mathrm{Hilb}(K3)$ [42]. We will return to this 4D/5D lift in Sect. 3.5. The conjecture has also been generalized to other $\mathcal{N}=4$ "CHL" models with different values of n_v in (21) [43, 44, 45]. More recently, the $Sp(4,\mathbb{Z})$ symmetry has been motivated by representing 1/4-BPS dyons as string networks on T^2, which lift to M2-branes with genus 2 topology [46]. Despite this suggestive interpretation, it is fair to say that the origin of $Sp(4)$ remains rather mysterious. In Sect. 7, we will formulate a similar conjecture, which relies on the 3-dimensional U-duality group $SO(8,24,\mathbb{Z})$ obtained by reduction on a thermal circle, rather than $Sp(4)$.

3 Special Geometry and Black Hole Attractors

In this section, we expose the formalism of special geometry, which governs the couplings of vector multiplets in $\mathcal{N}=2$, $D=4$ supergravity. We then derive the attractor flow equations, governing the radial evolution of the scalars in spherically BPS geometries. Finally, we illustrate these these constructions in the context of "very special" supergravity theories, which are simple toy models of $\mathcal{N}=2$ supergravity with symmetric moduli spaces. We follow the notations of [47], which gives a good overview of the essentials of special geometry. Useful reviews of the attractor mechanism include [48, 49, 50].

3.1 $\mathcal{N}$ = 2 SUGRA and Special Geometry

A general "ungauged" $\mathcal{N}=2$ supergravity theory in 4 dimensions may be obtained by combining massless supersymmetric multiplets with spin less or equal to 2:

(i) The gravity multiplet, containing the graviton $g_{\mu\nu}$, two gravitini ψ_μ^α, and one Abelian gauge field known as the graviphoton;
(ii) n_V vector multiplets, each consisting of one Abelian gauge field A_μ, two gaugini λ^α, and one complex scalar. The complex scalars z_i take values in a *projective special Kähler manifold* $\mathcal{M}_V$ of real dimension $2n_V$.
(iii) n_H hypermultiplets, each consisting of two complex scalars and two hyperinis $\psi, \tilde{\psi}$. The scalars take values in a *quaternionic-Kähler space* $\mathcal{M}_{\mathcal{H}}$ of real dimension $4n_H$.

Tensor multiplets are also possible, and can be dualized into hypermultiplets with special isometries. At two-derivative order, vector multiplets and hypermultiplets interact only gravitationally[4]. We will concentrate on the gravitational and vector multiplet sectors, which control the physics of charged BPS black holes. Nevertheless, we will encounter hypermultiplet moduli spaces in Sect. 7.3.1, when reducing the solutions to three dimensions.

The couplings of the vector multiplets, including the geometry of the scalar manifold $\mathcal{M}_V$, are conveniently described by means of a $Sp(2n_V+2)$ principal bundle $\mathcal{E}$ over $\mathcal{M}_V$ and its associated bundle $\mathcal{E}_V$ in the vector representation of $Sp(2n_V+2)$. The origin of the symplectic symmetry lies in electric-magnetic duality, which mixes the n_V vectors $\mathcal{A}_\mu$ and the graviphoton $\mathcal{A}_\mu$ together with their magnetic duals. Denoting a section Ω by its coordinates (X^I, F_I), the antisymmetric product

$$\langle \Omega, \Omega' \rangle = X^I F_I' - X'^I F_I \tag{27}$$

endows the fibers with a phase space structure, derived from the symplectic form $\langle d\Omega,\, d\Omega \rangle = dX^I \wedge dF_I$.

The geometry of the scalar manifold $\mathcal{M}_V$ is completely determined by a choice of a holomorphic section $\Omega(z) = (X^I(z), F_I(z))$ taking value in a Lagrangian cone, i.e. a dilation invariant subspace such that $dX^I \wedge dF_I = 0$. At generic points, one may express F_I in terms of their canonical conjugate X^I via a characteristic function $F(X^I)$ known as the *prepotential*:

[4] This is no longer true in "gauged" supergravities, where some of the hypermultiplets become charged under the vectors.

$$F_I = \frac{\partial F}{\partial X^I} , \qquad F\left(X^I\right) = \frac{1}{2} X^I F_I . \tag{28}$$

The second relation reflects the homogeneity of the Lagrangian and implies that F is an homogeneous function of degree 2 in the X^I. At generic points, the sections X^I $(I = 0 \dots n_V)$ may be chosen as *projective* holomorphic coordinates on $\mathscr{M}_V$ – equivalently, the n_V ratios $z^i = X^i/X^0$ $(i = 1 \dots n_V)$ may be taken as the holomorphic coordinates; these are known as (projective) special coordinates. Note however that a choice of F breaks manifest symplectic invariance, so special coordinates may not always be the most convenient ones.

Exercise 4. *Show that a symplectic transformation* $(X^I, F_I) \to (F_I, -X^I)$ *turns the prepotential into its Legendre transform.*

Once the holomorphic section $\Omega(z)$ is given, the metric on $\mathscr{M}_V$ is obtained from the Kähler potential

$$\mathscr{K}\left(z^i, \bar{z}^{\bar{i}}\right) = -\log K\left(X, \bar{X}\right) , \quad K(X, \bar{X}) = i\left(\bar{X}^I F_I - X^I \bar{F}_I\right) \tag{29}$$

This leads to a well-defined metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathscr{K}$, since under a holomorphic rescaling $\Omega \to e^{f(z^i)}\Omega$, $\mathscr{K} \to \mathscr{K} - f(z) - \bar{f}(\bar{z})$ changes by a Kähler transformation. Equivalently, Ω should be viewed as a section of $\mathscr{E}_V \otimes \mathscr{L}$ where $\mathscr{L}$ is the Hodge bundle over $\mathscr{M}_V$, namely, a line bundle whose curvature is equal to the Kähler form; its connection one-form is just $Q = (\partial_i \mathscr{K} dz^i - \partial_{\bar{i}} \mathscr{K} dz^{\bar{i}})/(2i)$. The rescaled section $\tilde{\Omega} = e^{\mathscr{K}/2}\Omega$ is then normalized to 1, and transforms by a phase under holomorphic rescalings of Ω. For later purposes, it will be convenient to introduce the derived section $U_i = D_i \tilde{\Omega} = (f_i^I,\, h_{iI})$ where

$$f_i^I = e^{\mathscr{K}/2} D_i X^I = e^{\mathscr{K}/2}\left(\partial_i X^I + \partial_i \mathscr{K}\, X^I\right) \tag{30}$$

$$h_{iI} = e^{\mathscr{K}/2} D_i F_I = e^{\mathscr{K}/2}\left(\partial_i F_I + \partial_i \mathscr{K}\, F_I\right) \tag{31}$$

The metric may thus be reexpressed as

$$g_{i\bar{j}} = -i \langle U_i, \bar{U}_{\bar{j}} \rangle = i\left(f_i^I \bar{h}_{\bar{j}I} - h_{iI} \bar{f}_{\bar{j}}^I\right) \tag{32}$$

After some algebra, one may show that the Riemann tensor on $\mathscr{M}_V$ takes the form

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - e^{2\mathscr{K}} C_{ikm} \bar{C}_{\bar{j}\bar{l}\bar{n}} g^{m\bar{n}} \tag{33}$$

where C_{ijk} is a holomorphic, totally symmetric tensor[5]

$$C_{ijk} = e^{-\mathscr{K}} \langle D_i U_j, U_k \rangle \tag{34}$$

[5] We follow the standard notation in the topological string literature, which differs from [47] by a factor of $e^{\mathscr{K}}$.

The foregoing formalism was in fact geared to produce a solution of (33), which embodies the constraint of supersymmetry and may be taken as the definition of a projective special Kähler manifold.

The kinetic terms of the n_V+1 Abelian gauge fields (including the graviphoton) may also be obtained from the holomorphic section Ω as

$$\begin{aligned}\mathscr{L}_{\rm Maxwell} &= -\,\mathrm{Im}\mathscr{N}_{IJ}\;\mathscr{F}^I\wedge\star\mathscr{F}^J+\mathrm{Re}\mathscr{N}_{IJ}\;\mathscr{F}^I\wedge\mathscr{F}^J\\ &=\mathrm{Im}\left[\bar{\mathscr{N}}_{IJ}\;\mathscr{F}^{I-}\wedge\star\;\mathscr{F}^{J-}\right]+\text{total der.}\end{aligned}\tag{35}$$

where $\mathscr{F}^{I-}=(\mathscr{F}^I-i\star\mathscr{F}^I)/2,\ I=0\dots n_V$ is the anti-self dual part of the field-strength, and $\mathscr{N}_{IJ}$ is defined by the relations

$$F_I=\mathscr{N}_{IJ}X^J\ ,\quad h_{iI}=\bar{\mathscr{N}}_{IJ}f_i^J\tag{36}$$

In terms of the prepotential F and its Hessian $\tau_{IJ}=\partial_I\partial_J F$,

$$\mathscr{N}_{IJ}=\bar{\tau}_{IJ}+2i\frac{(\mathrm{Im}\tau\cdot X)_I\;(\mathrm{Im}\tau\cdot X)_J}{X\cdot\mathrm{Im}\tau\cdot X}\tag{37}$$

While $\mathrm{Im}\tau_{IJ}$ has indefinite signature $(1,\ n_V)$, $\mathrm{Im}\mathscr{N}_{IJ}$ is a negative definite matrix, as required for the positive definiteness of the gauge kinetic terms in (35).

Exercise 5. *For later use, prove the relations*

$$\mathscr{K}=-\log\left[-2X^I[\mathrm{Im}\mathscr{N}]_{IJ}\bar{X}^J\right]\ ,\quad f_i^I[\mathrm{Im}\mathscr{N}]_{IJ}X^J=0\tag{38}$$

In order to study the invariance of (35) under electric-magnetic duality, it is useful to introduce the dual vector

$$\mathscr{G}_{I;\mu\nu}=\frac{1}{2}\frac{\partial\mathscr{L}_{\rm Maxwell}}{\partial\mathscr{F}^{I;\mu\nu}}=[\mathrm{Re}\mathscr{N}]_{IJ}\;\mathscr{F}^J+[\mathrm{Im}\mathscr{N}]_{IJ}\;\star\mathscr{F}^I\tag{39}$$

Under symplectic transformations, $\mathscr{N}$ transforms as a "period matrix" $\mathscr{N}\to(C+D\mathscr{N})(A+B\mathscr{N})^{-1}$, while the field strengths $(\mathscr{F}^{I-},\mathscr{G}_I^-=\bar{\mathscr{N}}_{IJ}\mathscr{F}^{J-}_{\mu\nu})$ transform as a symplectic vector, leaving (35) invariant. The electric and magnetic charges (p^I,q_I) are measured by the integral on a 2-sphere at spatial infinity of $(\mathscr{F}^{I-},\mathscr{G}_I^-)$ and transform as a symplectic vector too.

One linear combination of the n_V+1 field-strengths, the graviphoton

$$T^-_{\mu\nu}=-2i\,e^{\mathscr{K}/2}\,X^I\,[\mathrm{Im}\mathscr{N}]_{IJ}\mathscr{F}^{J-}_{\mu\nu}=e^{\mathscr{K}/2}\left(X^I\mathscr{G}_I^--F_I\mathscr{F}^{I-}\right)\tag{40}$$

plays a distinguished rôle, as its associated charge measured at infinity

$$Z=e^{\mathscr{K}/2}\,\left(q_IX^I-p^IF_I\right)\equiv e^{\mathscr{K}/2}W(X)\tag{41}$$

appears as the central charge in $\mathscr{N}=2$ supersymmetry algebra,

$$\left\{Q^i_\alpha,\bar{Q}_{\dot\alpha j}\right\}=P_\mu\sigma^\mu_{\alpha\dot\alpha}\delta^i_j\ ,\quad\left\{Q^i_\alpha,Q^j_\beta\right\}=Z\epsilon^{ij}\epsilon_{\alpha\beta}\tag{42}$$

In particular, there is a Bogomolony-Prasad-Sommerfeld (BPS) bound on the mass

$$M^2 \geq |Z|^2 \, m_P^2 \tag{43}$$

where m_P is the (duality invariant) 4-dimensional Planck scale, which is saturated when the state preserves 4 supersymmetries out of the 8 supersymmetries of the vacuum.

3.2 $\mathscr{N}$ = 2 SUGRA and String Theory

There are several ways to obtain $\mathscr{N}=2$ supergravities in 4 dimensions from string theory. Type IIB string compactified on a Calabi-Yau three-fold Y leads to $\mathscr{N}=2$ supergravity with $n_V = h_{2,1}(Y)$ vector multiplets and $n_H = h_{1,1}(Y)+1$ hypermultiplets. The scalars in $\mathscr{M}_V$ parameterize the complex structure of the Calabi-Yau metric on Y. The associated vector fields are the reduction of the 10D Ramond-Ramond 4-form on the various 3-cycles in $H_3(Y,\mathbb{R})$. The holomorphic section Ω is then given by the periods of the holomorphic 3-form Ω (abusing the notation) on a symplectic basis (A^I, B_I) of $H_3(Y,\mathbb{R})$:

$$X^I = \int_{A^I} \Omega\,, \quad F_I = \int_{B^I} \Omega \tag{44}$$

The Kähler potential on the moduli of complex structures is just

$$\mathscr{K} = -\log\left[i \int_Y \Omega \wedge \bar{\Omega} \right] \tag{45}$$

which agrees with (29) by Riemann's bilinear identity. As we shall see later, it is determined purely at tree-level and can be computed purely in field theory. The central charge of a state with electric-magnetic charges p^I, q_I may be rewritten as

$$Z = \frac{\int_\gamma \Omega}{\sqrt{i \int_Y \Omega \wedge \bar{\Omega}}} \tag{46}$$

where $\gamma = q_I A^I - p^I B_I$ and is recognized as the mass of a D3-brane wrapped on a special Lagrangian 3-cycle $\gamma \in H_3(Y,\mathbb{Z})$.

On the other hand, the scalars in $\mathscr{M}_H$ parameterize the complexified Kähler structure of Y, the fluxes (or more appropriately, Wilson lines) of the Ramond-Ramond two-forms along $H_{\text{even}}(Y,\mathbb{R})$, as well as the axio-dilaton. The axio-dilaton, zero and six-form RR potentials form a "universal hypermultiplet" sector inside $\mathscr{M}_H$. In contrast to the vector-multiplet metric, the hyper-multiplet metric receives one-loop and non-perturbative corrections from Euclidean D-branes and NS-branes wrapped on $H_{\text{even}}(Y)$.

The situation in type IIA string compactified on a Calabi-Yau three-fold $\tilde{Y}$ is reversed: the vector-multiplet moduli space describes the complexified Kähler structure of $\tilde{Y}$, while the hypermultiplet moduli space describes its complex structure, together with the Wilson lines of the Ramond-Ramond forms along $H_{\rm odd}(\tilde{Y})$ and the axio-dilaton. As in IIB, the vector-multiplet moduli space is determined at tree-level only, but receives α' corrections. Letting $J = B_{NS} + i\omega_K$ be the complexified Kähler form, γ^A be a basis of $H_{1,1}(\tilde{Y},\mathbb{Z})$, and γ_A the dual basis of $H_{2,2}(\tilde{Y},\mathbb{Z})$, the holomorphic section Ω (not to be confused with the holomorphic three-form on $\tilde{Y}$) is determined projectively by the special coordinates

$$X^A/X^0 = \int_{\gamma^A} J\,, \quad F_A/X^0 = \int_{\gamma_A} J\wedge J \tag{47}$$

In the limit of large volume, the Kähler potential (in the gauge $X^0 = 1$) is given by the volume in string units,

$$\mathscr{K} = -\log \int_{\tilde{Y}} J\wedge J\wedge J \tag{48}$$

originating from the cubic prepotential

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} + \dots \tag{49}$$

Here, C_{ABC} are the intersection numbers of the 4-cycles $\gamma_{A,B,C}$. At finite volume, there are corrections to (49) from worldsheet instantons wrapping effective curves in $H_2^+(\tilde{Y},\mathbb{Z})$, to which we will return in Sect. 4.3. The central charge following from (49) is

$$Z = e^{\mathscr{K}/2} X^0 \left(q_0 + q_A \int_{\gamma^A} J - p^A \int_{\gamma_A} J\wedge J - p^0 \int_{\tilde{Y}} J\wedge J\wedge J \right) \tag{50}$$

so that q_0, q_A, p^A, p^0 can be identified as the D0,D2,D4 and D6 brane charge, respectively.

While (49) expresses the complete prepotential in terms of the geometry of $\tilde{Y}$, the most practical way of computing it is to use mirror symmetry, which relates type IIA compactified on $\tilde{Y}$ to type IIB compactified on Y, where $(Y,\tilde{Y})$ form a "mirror pair"; this implies in particular that $h_{1,1}(Y) = h_{2,1}(\tilde{Y})$ and $h_{1,1}(\tilde{Y}) = h_{2,1}(Y)$ (see [51] for a review).

On the other hand, the tree-level metric on the hypermultiplet moduli space $\mathscr{M}_H$ in type IIA compactified on $\tilde{Y}$ may be obtained from the vector-multiplet metric $\mathscr{M}_V$ in type IIB compactified on the *same* Calabi-Yau $\tilde{Y}$, by compactifying on a circle S^1 to 3 dimensions, T-dualizing along S^1 and decompactifying back to 4 dimensions. We shall return to this "c-map" procedure in Sect. 7.3.1.

Finally, another way to obtain $\mathscr{N} = 2$ supergravity in 4 dimensions is to compactify the heterotic string on $K3\times T^2$. Since the heterotic axio-dilaton is now a vector-multiplet, $\mathscr{M}_V$ now receives loop and instanton corrections, while $\mathscr{M}_H$ is determined purely at tree-level (albeit with α' corrections).

3.3 Attractor Flows and Bekenstein-Hawking Entropy

We now turn to static, spherically symmetric BPS black hole solutions of $\mathcal{N}=2$ supergravity. The assumed isometries lead to the metric ansatz

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left(dr^2 + r^2 d\Omega_2^2\right) \tag{51}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta\ d\phi^2$ is the round metric on S^2, and U depends on r only. We took advantage of the BPS property to restrict to flat 3D spatial slices[6]. Moreover, the scalars z^i in the vector multiplet moduli space are taken to depend on r only. The gauge fields are uniquely determined by the equations of motion and Bianchi identities:

$$\mathscr{F}^{I-} = \frac{1}{2}\left[p^I - i[\mathrm{Im}\mathcal{N}]^{IJ}\left(q_J - [\mathrm{Re}\mathcal{N}]_{JK}p^K\right)\right]\cdot\left[\sin\theta\ d\theta\wedge d\phi - i\frac{e^{2U}}{r^2}dt\wedge dr\right] \tag{52}$$

where (p^I, q_I) are the magnetic and electric charges, and $[\mathrm{Im}\mathcal{N}]^{IJ} = [\mathrm{Im}\mathcal{N}]^{-1}_{IJ}$.

Assuming that the solution preserves half of the 8 supersymmetries, the gravitino and gaugino variations lead to a set of first-order equations [49, 53, 54, 55][7]

$$r^2\frac{dU}{dr} = |Z|\ e^U \tag{53}$$

$$r^2\frac{dz^i}{dr} = 2\ e^U\ g^{i\bar{j}}\partial_{\bar{j}}|Z| \tag{54}$$

where Z is the central charge defined in (41). These equations govern the radial evolution of U and $z^i(r)$, and are usually referred to as "attractor flow equations", for reasons which will become clear shortly. The boundary conditions are such that $U(r\to\infty)\to 0$ at spatial infinity, while the vector multiplet scalars z^i go to their vacuum values z^i_∞. The black hole horizon is reached when the time component of the metric $g_{tt} = e^{2U}$ vanishes, i.e. at $U = -\infty$.

Defining $\mu = e^{-U}$ so that $r^2 d\mu/dr = -|Z|$, the second equation may be cast in the form of a gradient flow, or RG flow,

$$\mu\frac{dz^i}{d\mu} = -g^{i\bar{j}}\partial_{\bar{j}}\log|Z|^2 \tag{55}$$

As a consequence, $|Z|$ decreases from spatial infinity, where $\mu = 1$, to the black hole horizon, when $\mu\to+\infty$. The scalars z^i therefore settle to values $z^i_*(p,q)$ which minimize the BPS mass $|Z|$; in particular, the vector multiplet scalars are "attracted" to a

[6] This condition may be relaxed, if one allows for a non-trivial profile of the hypermultiplets [52].

[7] We shall provide a full derivation of (53), (54) in Sect. 7, but for now we accept them and proceed with their consequences.

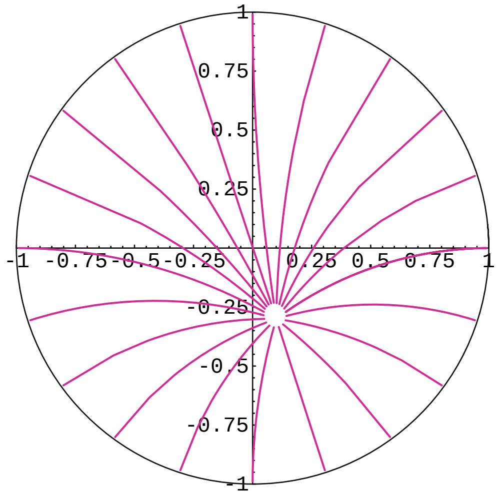

Fig. 2 Radial flow for the Gaussian one-scalar model, for charges $(p^0, p^1, q_1, q_0) = (4, 1, 1, 2)$. All trajectories are attracted to $z_* = X^1/X^0 = (1-3i)/10$ at $r = 0$

fixed value at the horizon, independent[8] of the asymptotic values z^i_∞, and determined only by the charges (p^I, q_I). This attractor behavior is illustrated in Fig. 2 for the case of the Gaussian one-scalar model with prepotential $F = -i[(X^0)^2 - X^1)^2]/2$, whose moduli space corresponds to the Poincaré disk $|z| < 1$. It should be noted that the attractor behavior is in fact a consequence of extremality rather than supersymmetry, as was first recognized in [55].

We shall assume that the charges (p^I, q_I) are chosen such that at the attractor point, $Z = Z_* \neq 0$, since otherwise the solution becomes singular. Equation (53) may be easily integrated near the horizon,

$$\mu = e^{-U} \sim |Z_*|/r \tag{56}$$

Defining $z = |Z_*|^2/r$, it is easy to see that the near-horizon metric becomes $AdS_2 \times S^2$, as in (13), where the prefactor $(p^2 + q^2)$ is replaced by $|Z_*|^2$. The Bekenstein-Hawking entropy is one quarter of the horizon area,

$$S_{BH} = \frac{1}{4} \cdot 4\pi \lim_{r \to 0} e^{-2U} r^2 = \pi |Z_*|^2 \tag{57}$$

This is a function of the electric and magnetic charges only, by virtue of the attractor mechanism, except for possible discrete labels (or "area codes") corresponding to different basins of attraction.

We shall now put these results in a more manageable form, by making use of some special geometry identities discussed in Sect. 3. First, using the derived section $U_i = (f_i^I, h_{iI})$ defined in (30) and the property (36), one easily finds

[8] In some cases, there can exist different basins of attraction, leading to a discrete set of possible values $z^i_*(p,q)$ for a given choice of charges. This is typically connected with the "split attractor flow" phenomenon [56].

$$\partial_i Z = f_i^I (q_I - \bar{\mathcal{N}}_{IJ} p^J) - \frac{1}{2} Z \partial_i \mathcal{K} \ , \quad \partial_{\bar{i}} Z = \frac{1}{2} Z \partial_{\bar{i}} \mathcal{K} \tag{58}$$

so that

$$\frac{\partial_i |Z|}{|Z|} = \frac{1}{2} \left(\frac{\partial_i Z}{Z} + \frac{\partial_i \bar{Z}}{\bar{Z}} \right) = \frac{1}{Z} f_i^I \left(q_I - \bar{\mathcal{N}}_{IJ} p^J \right) \tag{59}$$

This allows to rewrite (54) as

$$r^2 \frac{dz^i}{dr} = -\sqrt{\frac{Z}{\bar{Z}}} e^U g^{i\bar{j}} \bar{f}_{\bar{j}}^J \left(q_I - \mathcal{N}_{IJ} p^J \right) \tag{60}$$

The stationary value of z^i at the horizon is thus obtained by setting the right-hand side of this equation to zero, i.e.

$$f_i^J \left(q_I - \bar{\mathcal{N}}_{IJ} p^J \right) = 0 \tag{61}$$

The rectangular matrix f_i^I has a unique zero eigenvector, given by the second equality in (38). Hence, (61) implies

$$q_I - \bar{\mathcal{N}}_{IJ} p^J = C \, \mathrm{Im} \mathcal{N}_{IJ} X^J \tag{62}$$

Contracting either side with $\bar{X}^I$ and using the first equation in (38) allows to compute the value of α,

$$C = -2\bar{Z} \, e^{\mathcal{K}/2} \tag{63}$$

Moreover, using again (36), one may rewrite (62) and its complex conjugate, equivalently as two real equations

$$p^I = \mathrm{Im} \left(C X^I \right) \ , \quad q_I = \mathrm{Im} \left(C F_I \right) \tag{64}$$

while the Bekenstein-Hawking entropy (57) is given by

$$S_{BH} = \frac{\pi}{4} |C|^2 e^{-\mathcal{K}(X,\bar{X})} = \frac{i\pi}{4} |C|^2 \left(\bar{X}^I F_I - X^I \bar{F}_I \right) \tag{65}$$

Making use of the fact that near the horizon, $e^{-U} \sim |Z_*|/r$, it is convenient to rescale the holomorphic section $\Omega = (X^I, \, F_I)$ into

$$\begin{pmatrix} Y^I \\ G_I \end{pmatrix} = 2i \, r \, e^{\frac{1}{2}\mathcal{K}(X,\bar{X}) - U} \sqrt{\frac{\bar{Z}}{Z}} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{66}$$

in such a way that

$$e^{-\mathcal{K}(Y,\bar{Y})} = 4 r^2 e^{-2U} \ , \quad \arg W(Y) = \pi/2 \tag{67}$$

where we defined, in line with (29) and (41),

$$K(Y,\bar{Y}) = \left[i \left(\bar{Y}^I G_I - Y^I \bar{G}_I \right) \right] = e^{-\mathcal{K}(Y,\bar{Y})} \ , \quad W(Y) = q_I Y^I - p^I G_I \tag{68}$$

In this fashion, we have incorporated the geometric variable U into the symplectic section $(Y^I,\, G_I)$ and fixed the phase. In this new "gauge"[9], which amounts to setting $C\equiv i$, (64) and (57) simplify into

$$\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \mathrm{Re} \begin{pmatrix} Y^I \\ G_I \end{pmatrix} \tag{69}$$

$$S_{BH} = \frac{\pi}{4} K(Y,\bar{Y}) = \frac{i\pi}{4}\left[\bar{Y}^I G_I - Y^I \bar{G}_I\right] \tag{70}$$

These equations, some times known as "stabilization equations", are the most convenient way of summarizing the endpoint of the attractor mechanism, as will become apparent in the next subsection.

3.4 Bekenstein-Hawking Entropy and Legendre Transform

A key observation for later developments is that the Bekenstein-Hawking entropy (69) is simply related by Legendre transform[10] to the tree-level prepotential F. To see this, note that the first equation in (69) is trivially solved by setting $Y^I = p^I + i\phi^I$, where ϕ^I is real. The entropy is then rewritten as

$$S_{BH} = \frac{i\pi}{4}\left[(Y^I - 2i\phi^I)G_I - (\bar{Y}^I + 2i\phi^I)\bar{G}_I\right] \tag{71}$$

$$= \frac{i\pi}{2}\left[F(Y) - \bar{F}(\bar{Y})\right] + \frac{\pi}{2}\phi^I\left[G_I + \bar{G}_I\right] \tag{72}$$

where, in going from the second to the third line, we used the homogeneity of the prepotential, $Y^I G_I = 2F(Y)$. On the other hand, the second stabilization equation yields

$$q_I = \frac{1}{2}\left(G_I + \bar{G}_I\right) = \frac{1}{2i}\left(\frac{\partial F}{\partial \phi^I} - \frac{\partial \bar{F}}{\partial \phi^I}\right) \tag{73}$$

Thus, defining

$$\mathscr{F}(p^I,\phi^I) = -\pi\ \mathrm{Im}\left[F(p^I + i\phi^I)\right] \tag{74}$$

the last equation in (72) becomes

$$S_{BH}(p^I,q_I) = \langle \mathscr{F}(p^I,\phi^I) + \pi\ \phi^I q_I \rangle_{\phi^I} \tag{75}$$

where the right-hand side is evaluated at its extremal value with respect to ϕ^I. In usual thermodynamical terms, this implies that $\mathscr{F}(p^I,\ \phi^I)$ should be viewed as the free energy of an ensemble of black holes in which the magnetic charge p^I is fixed,

[9] This is an abuse of language, since the scale factor is a priori not a holomorphic function of z^i.

[10] This was first observed in [57] and spelled out more clearly in [4].

but the electric charge q_I is free to fluctuate at an electric potential $\pi\phi^I$. The implications of this simple observation will be profound in Sect. 5.3, when we discuss the higher-derivative corrections to the Bekenstein-Hawking entropy.

Exercise 6. *Apply this formalism to show that the entropy of a D0-D4 bound state in type IIA string theory compactified on a Calabi-Yau three-fold, in the large charge regime, is given by*

$$S_{BH} = 2\pi\sqrt{-C_{ABC}p^A p^B p^C q_0} \tag{76}$$

and compare to (20).

Exercise 7. *Show that the Bekenstein-Hawking entropy (70) can be obtained by extremizing*

$$\Sigma_{p,q}(Y,\bar{Y}) = -\frac{\pi}{4}\left[K(Y,\bar{Y}) + 2i[W(Y) - \bar{W}(\bar{Y})]\right] \tag{77}$$

with respect to Y, $\bar{Y}$*, where* $K(Y,\bar{Y})$ *and* $W(Y)$ *are defined in* (68) *[7, 58]. Observe that* (75) *is recovered by extremizing over* $\mathrm{Re}(Y)$.

Exercise 8. *Define the Hesse potential* $\Sigma(\phi^I, \chi_I)$ *as the Legendre transform of the topological free energy with respect to the magnetic charges* p^I*,*

$$\Sigma\left(\phi^I, \chi_I\right) = \left\langle \mathscr{F}\left(p^I, \phi_I\right) + \pi\chi_I p^I \right\rangle_{p^I} \tag{78}$$

Show that the dependence of Σ *on the electric and magnetic potentials* (ϕ^I, χ_I) *is identical (up to a sign) to that of the black hole entropy* S_{BH} *on the charges* (p^I, q_I)*. Compare to* $\Sigma_{p,q}$ *in the previous Exercise.*

3.5 Very Special Supergravities and Jordan Algebras

In the remainder of this section, we illustrate the previous results on a special class of $\mathcal{N}=2$ supergravities, whose vector-multiplet moduli spaces are given by symmetric spaces. These are interesting toy models, which arise in various truncations of string compactifications. Moreover, they are related to by analytic continuation to $\mathcal{N}>2$ theories, which will be further discussed in Sect. 7.

The simplest way to construct these models is to start from 5 dimensions [59]: the vector multiplets consist of one real scalar for each vector, and their couplings are given by

$$S = \int d^5x\,\sqrt{-g}\,\left(R - G_{ij}\partial_\mu\phi^i\partial_\mu\phi^j\right) - \mathring{a}_{AB}\,F^A\wedge\star F^B + \frac{1}{24}\int C_{ABC}\,A^A\wedge F^B\wedge F^C \tag{79}$$

where the Chern-Simons-type couplings C_{ABC} are constant, for gauge invariance. $\mathcal{N}=2$ supersymmetry requires the real scalar fields ϕ^i to take value in the cubic hypersurface $\mathcal{M}_5 = \{\xi,\, N(\xi)=1\}$ in an ambient space $\xi\in\mathbb{R}^{n_V+1}$, where

$$N(\xi) = \frac{1}{6} C_{ABC}\, \xi^A\, \xi^B\, \xi^C \tag{80}$$

The metric G_{ij} is then the pull-back of the ambient space metric $a_{AB} d\xi^A d\xi^B$ to $\mathscr{M}_5$, where

$$a_{AB} = -\frac{1}{2} \partial_{\xi^A} \partial_{\xi^B} N(\xi) \tag{81}$$

The gauge couplings $\mathring{a}_{AB}$ are instead given by the restriction of a_{AB} to the hypersurface $\mathscr{M}_5$. Upon reduction from 5 dimensions to 4 dimensions, using the standard Kaluza-Klein ansatz

$$ds_5^2 = e^{2\sigma}(dy + B_\mu dx^\mu)^2 + e^{-\sigma} g_{\mu\nu} dx^\mu dx^\nu \tag{82}$$

the Kaluza-Klein gauge field B_μ provides the graviphoton, while the constraint $N(\xi) = 1$ is relaxed to $N(\xi) = e^{3\sigma}$. Moreover, ξ^A combine with the fifth components a^A of the gauge fields A^A into complex scalars $t^A = a^A + i\xi^A = X^A/X^0$, which are the special coordinates of a special Kähler manifold $\mathscr{M}_4$ with prepotential

$$F = N(X^A)/X^0 \tag{83}$$

In general, neither $\mathscr{M}_5$ nor $\mathscr{M}_4$ are symmetric spaces. The conditions for $\mathscr{M}_5$ to be a symmetric space were analyzed in [59] and found to have a remarkably simple interpretation in terms of Jordan algebras: these are commutative, non-associative algebras J satisfying the "Jordan identity"

$$x \circ (y \circ x^2) = (x \circ y) \cdot x^2 \tag{84}$$

where $x^2 = x \circ x$ (see e.g. [60] for a nice review).

Exercise 9. *Show that the algebra of $n \times n$ hermitean matrices with product $A \circ B = \frac{1}{2}(AB + BA)$ is a Jordan algebra.*

Jordan algebras were introduced and completely classified in [61] in an attempt to generalize quantum mechanics beyond the field of complex numbers. The ones relevant here are those which admit a norm N of degree 3 – rather than giving the axioms of the norm, we shall merely list the allowed possibilities:

(i) One trivial case: $J = \mathbb{R}$, $N(\xi) = \xi^3$
(ii) One infinite series: $J = \mathbb{R} \oplus \Gamma$ where Γ is the Clifford algebra of $O(1, n-1)$, $N(\xi \oplus \gamma) = \xi \gamma^a \gamma^b \eta_{ab}$
(iii) Four exceptional cases: $J = \mathrm{Herm}_3(\mathbb{D})$, the algebra of 3×3 hermitean matrices $\xi = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix}$ where α_i are real and x_i are in one of the four "division algebras" $\mathbb{D} = \mathbb{R}, \mathbb{C}$, the quaternions $\mathbb{H}$ or octonions $\mathbb{O}$. In each of these cases, the cubic norm is the "determinant" of ξ

Table 1 Invariance groups associated to degree 3 Jordan algebras. The lower 4×4 part is known as the "Magic Square", due to its symmetry along the diagonal [62]

J	Aut(J)	$\mathrm{Str}_0(J)$	Conf(J)	QConf(J)
$\mathbb{R}$	1	1	$Sl(2,\,\mathbb{R})$	$G_{2(2)}$
$\mathbb{R}\oplus\Gamma_{n-1,\,1}$	$SO(n-1)$	$SO(n-1,\,1)$	$Sl(2)\times SO(n,\,2)$	$SO(n+2,\,4)$
$J_3^{\mathbb{R}}$	$SO(3)$	$Sl(3,\,\mathbb{R})$	$Sp(6)$	$F_{4(4)}$
$J_3^{\mathbb{C}}$	$SU(3)$	$Sl(3,\,\mathbb{C})$	$SU(3,\,3)$	$E_{6(+2)}$
$J_3^{\mathbb{H}}$	$USp(6)$	$SU^*(6)$	$SO^*(12)$	$E_{7(-5)}$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$	$E_{8(-24)}$

$$N(\xi)=\alpha_1\alpha_2\alpha_3-\alpha_1 x_1\bar{x}_1-\alpha_2 x_2\bar{x}_2-\alpha_3 x_3\bar{x}_3+2\mathrm{Re}(x_1x_2x_3) \tag{85}$$

For $J_3^{\mathbb{C}}$, this is equivalent to the determinant of an unconstrained 3×3 real matrix, and for $J_3^{\mathbb{H}}$ to the Pfaffian of a 6×6 antisymmetric matrix.

To each of these Jordan algebras, one may attach several invariance groups, summarized in Table 1:

(a) Aut(J), the group of automorphisms of J, which leaves invariant the structure constants of the Jordan product;
(b) Str(J), the "structure" group, which leaves invariant the norm $N(\xi)$ up to a rescaling; and the "reduced structure group" $\mathrm{Str}_0(J)$, where the center has been divided out;
(c) Conf(J), the "conformal" group, such that the norm of the difference of two elements $N(\xi-\xi')$ is multiplied by a product $f(\xi)f(\xi')$; as a result, the "cubic light-cone" $N(\xi-\xi')=0$ is invariant;
(d) QConf(J), the "quasi-conformal group", which we will describe in Sect. 7.5.

In the case ii) above, Aut(J), Str(J) and Conf(J) are just the orthogonal group $SO(n-1)$, Lorentz group $SO(n-1,1)$ and conformal group $SO(n,\,2)$ times an extra $Sl(2)$ factor.

The relevance of these groups for physics is as follows: choosing $N(\xi)$ in (80) to be equal to the norm form of a Jordan algebra J, the vector-multiplet moduli spaces for the resulting $\mathcal{N}=2$ supergravity in $D=5$ and $D=4$ are symmetric spaces

$$\mathcal{M}_5=\frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)}\,,\quad \mathcal{M}_4=\frac{\mathrm{Conf}(J)}{\widetilde{\mathrm{Str}}_0(J)\times U(1)}\,, \tag{86}$$

where $\widetilde{\mathrm{Str}}_0(J)$ denotes the compact real form of $\mathrm{Str}_0(J)$. In either case, the group in the denominator is the maximal subgroup of the one in the numerator, which guarantees that the quotient has positive definite signature. The resulting spaces are shown in Table 2, together with the ones which appear upon reduction to $D=3$ on a space-like and time-like direction respectively, to be discussed in Sect. 7.5 below. The first column indicates the number of supercharges in the corresponding supergravity: the

Table 2 Moduli spaces for supergravities with symmetric moduli spaces. The last column refers to the reduction from 4 dimensions to 3 along a time-like direction, which will become relevant in Sect. 7

Q	J	$D=5$	$D=4$	$D=3$	$D=3^*$
8			$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times Sl(2)\times U(1)}$
8	$\Gamma_{n-1,1}$	$\mathbb{R}\times\frac{SO(n-1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
8			$\frac{Sl(2)}{U(1)}$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\frac{SU(2,1)}{Sl(2)\times U(1)}$
8	$\mathbb{R}$	$\varnothing$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
8	$J_3^{\mathbb{R}}$	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times Sl(2)}$
8	$J_3^{\mathbb{C}}$	$\frac{Sl(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(+2)}}{SU(3,3)\times Sl(2)}$
24	$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times Sl(2)}$
8	$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times Sl(2)}$
10				$\frac{Sp(2n,4)}{Sp(2n)\times Sp(4)}$	
12				$\frac{SU(n,4)}{SU(n)\times SU(4)}$	

Table 2 (continued)

Q	J	$D=5$	$D=4$	$D=3$	$D=3^*$
16	$\Gamma_{n-5,5}$	$\mathbb{R}\times\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}$	$\frac{Sl(2)}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,2)}$
18				$\frac{F_{4(-20)}}{SO(9)}$	
20	$M_{1,2}(\mathbb{O})$		$\frac{SU(5,1)}{SU(5)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$	$\frac{E_{6(-14)}}{SO^*(10)\times SO(2)}$
32	$J_3^{\mathbb{O}_s}$	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$

above discussion applies strictly speaking to cases with 8 supercharges (i.e. $\mathcal{N}=2$ supersymmetry in 4 dimensions), but other cases can also be reached with similar techniques, using different real forms of the Jordan algebras above[11].

The $\mathrm{Str}_0(J)$ invariance of the metric on $\mathcal{M}_5$ is indeed obvious from (81) above. The $\mathrm{Conf}(J)$ invariance of the metric on the special Kähler space $\mathcal{M}_4$ is manifest too, since the Kähler potential following from (83) is proportional to the log of the "cubic light-cone",

$$\mathcal{K}(z,\bar{z}) = -\log N(z_i - \bar{z}_i) \ , \tag{87}$$

invariant under $\mathrm{Conf}(J)$ up to Kähler transformations. Such special Kähler spaces are known as hermitean symmetric tube domains and are higher dimensional analogues of Poincaré's upper half plane.

It should be pointed out that there also exist $D=4$ SUGRAs with symmetric moduli space which do not descend from 5 dimensions: they may be described by a generalization of Jordan algebras known as "Freudenthal triple systems", but we will not discuss them in any detail here. Similarly, there exist $D=3$ supergravity theories with symmetric moduli spaces which cannot be lifted to 4 dimensions.

In general, it is not known whether these very special supergravities arise as the low-energy limit of string theory. All except the exceptional $J_3^{\mathbb{O}}$ case can be obtained formally by truncation of $\mathcal{N}=8$ supergravity, but it is in general unclear how to consistently enforce this truncation. A notable exception is the case based on $J=\Gamma_{9,1}$, which is realized in type IIA string theory compactified on a freely acting orbifold of $K3\times T^2$, or a CHL orbifold of the heterotic string on T^6 [64]. The model with $J=J_3^{\mathbb{C}}$ arises in the untwisted sector of type IIA compactified on the "Z-manifold" $T^6/\mathbb{Z}_3$ [65], but there are also massless fields from the twisted sector. We shall mostly use these theories at toy models in the sequel and assume that discrete subgroups $\mathrm{Str}_0(J,\mathbb{Z})$ and $\mathrm{Conf}(J,\mathbb{Z})$ remain as quantum symmetries of the full quantum theory, if it exists.

3.6 Bekenstein-Hawking Entropy in Very Special Supergravities

As an illustration of the simplicity of these models, we shall now proceed and compute the Bekenstein-Hawking entropy for BPS black holes with arbitrary charges, following [8]. A key property which renders the computation tractable is the fact that the prepotential (83) obtained from any Jordan algebra is invariant (up to a sign) under Legendre transform in all variables, namely,

$$\langle N(X^A)/X^0 + X^A Y_A + X^0 Y_0 \rangle_{X^I} = -N(Y)/Y^0 \tag{88}$$

[11] For example, the cubic invariant of $E_{6(6)}$ appearing in $\mathcal{N}=8$ supergravity can be obtained from (85) by replacing the usual octonions $\mathbb{O}$ by the split octions $\mathbb{O}_s$, whose norm $x\bar{x}$ has split signature (4,4); see [63] for a recent discussion.

Exercise 10. *Show that* (88) *is equivalent to the "adjoint identity" for Jordan algebras,* $X^{\sharp\sharp} = N(X)X$ *where* $X^\sharp_A = \frac{1}{2}C_{ABC}X^B X^C$ *is the "quadratic map" from* J *to its dual.*

In fact, just imposing (88) leads to the same classification (i),(ii),(iii) as above. This was shown independently in [66], as a first step in finding cubic analogues of the Gaussian, invariant under Fourier transform (see [67] for a short account).

Exercise 11. *Check by explicit computation that for the "STU" model,* $(1/X^0)\, e^{N(X^A)/X^0}$ *is invariant under Fourier transform, namely,*

$$\int \frac{dX^0 dX^1 dX^2 dX^3}{X^0} \exp\left[i\frac{X^1X^2X^3}{\hbar X^0} + iX^I Y_I \right] = \frac{\hbar}{Y^0} \exp\left[i\hbar \frac{Y_1Y_2Y_3}{Y_0} \right] \tag{89}$$

Conclude that the semi-classical approximation to this integral is exact. Hint: perform the integral over X^1, X^2, X^0, X^3 *in this order.*

In order to compute the Bekenstein-Hawking entropy, we start from the "free energy" (74)

$$\mathscr{F}(p,\phi) = \frac{\pi}{(p^0)^2 + (\phi^0)^2} \left\{ p^0 \left[\phi^A p^\sharp_A - N(\phi) \right] + \phi^0 \left[p^A \phi^\sharp_A - N(p) \right] \right\} \tag{90}$$

To eliminate the quadratic term in ϕ^A, let us change variables to

$$x^A = \phi^A - \frac{\phi^0}{p^0} p^A \ , \quad x^0 = [(p^0)^2 + (\phi^0)^2]/p^0 \tag{91}$$

Moreover, we introduce an auxiliary variable t, such that upon eliminating t we recover (90):

$$S_{BH} = \pi \left\langle -\frac{N(x^A)}{x^0} + \frac{p^\sharp_A + p^0 q_A}{p^0} x^A - \frac{t}{4}\left(\frac{x^0}{p^0} - 1 \right) - \frac{\left(2N(p) + p^0 p^I q_I\right)^2}{t\ (p^0)^2} \right\rangle_{\{x^I, t\}} \tag{92}$$

Extremizing over x^I now amounts to Legendre transforming $N(x)/x^0$, which according to (88) reproduces $-N(y)/y^0$ where y^I are the coefficients of the linear terms in x^I, so

$$S_{BH} = \pi \left\langle 4\frac{N[p^\sharp_A + p^0 q_A]}{(p^0)^2 t} - \frac{[2N(p) + p^0 p^I q_I]^2}{t\ (p^0)^2} + \frac{t}{4} \right\rangle_t \tag{93}$$

Finally, extremizing over t leads to

$$S_{BH} = \frac{\pi}{p^0} \sqrt{4N[p^\sharp_A + p^0 q_A] - [2N(p) + p^0 p^I q_I]^2} \tag{94}$$

The pole at $p^0 = 0$ is fake: Upon Taylor expanding $N[p^\sharp_A + p^0 q_A]$ in the numerator and further using the homogeneity of N, its coefficient cancels. The final result gives the entropy as the square root of a quartic polynomial in the charges,

$$S_{BH} = \pi \sqrt{I_4(p^I, q_I)} \tag{95}$$

where

$$I_4\left(p^I, q_I\right) = 4p^0 N\left(q_A\right) - 4q_0 N\left(p^A\right) + 4q_\sharp^A p_A^\sharp - \left(p^0 q_0 + p^A q_A\right)^2 \tag{96}$$

The fact that this quartic polynomial is invariant under the linear action of the four-dimensional "U-duality" group Conf(J) on the symplectic vector of charges $(p^I,\ q_I)$ follows from Freudenthal's "triple system construction". Several examples are worth mentioning:

- For the "STU" model with $N(\xi) = \xi^1 \xi^2 \xi^3$, the electric-magnetic charges transform as a (2,2,2) of Conf$(J) = Sl(2)^3$, so it can be viewed as sitting at the 8 corners of a cube; the quartic invariant is known as Cayley's "hyperdeterminant"

$$I_4 = -\frac{1}{2} \epsilon^{AB} \epsilon^{CD} \epsilon^{ab} \epsilon^{cd} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} q_{Aa\alpha} Q_{Bb\beta} Q_{Cc\gamma} Q_{Dd\delta} \tag{97}$$

  This has recently been related to the "three-bit entanglement" in quantum information theory[12] [68, 69, 70].
- More generally, for the infinite series, where the charges transform as a $(2,\ n)$ of $Sl(2) \times SO(2,n)$, the quartic invariant is

$$I_4 = (\vec{q}_e \cdot \vec{q}_e)(\vec{q}_m \cdot \vec{q}_m) - (\vec{q}_e \cdot \vec{q}_m)^2 \tag{98}$$

  Up to a change of signature of the orthogonal group, this is the quartic invariant which appears in the entropy of 1/4-BPS black holes in $\mathscr{N} = 4$ theories (22).
- In the exceptional $J_3^{\mathbb{O}}$ case, I_4 is the quartic invariant of the 56 representation of $E_{7(-25)}$. Replacing $\mathbb{O}$ by the split octonions $\mathbb{O}_s$, one obtains the quartic invariant of $E_{7(7)}$, which appears in the entropy $S = \pi\sqrt{I_4}$ of 1/8-BPS states in $\mathscr{N} = 8$ supergravity [71],

$$I_4(P,Q) = -\mathrm{Tr}(QPQP) + \frac{1}{4}\left(\mathrm{Tr}QP\right)^2 - 4\left[\mathrm{Pf}(P) + \mathrm{Pf}(Q)\right] \tag{99}$$

  where the entries in the antisymmetric 8×8 matrices Q and P may be identified as [8]:

$$Q = \begin{pmatrix} [D2]^{ij} & [F1]^i & [kkm]^i \\ -[F1]^i & 0 & [D6] \\ -[kkm]^i & -[D6] & 0 \end{pmatrix}, \quad P = \begin{pmatrix} [D4]_{ij} & [NS5]_i & [kk]_i \\ -[NS5]_i & 0 & [D0] \\ -[kk]_i & -[D0] & 0 \end{pmatrix}, \tag{100}$$

[12] According to Freudenthal's construction, the electric and magnetic charges naturally arrange themselves into a square (rather than a cube) $\begin{pmatrix} p^0 & p^I \\ q_I & q_0 \end{pmatrix}$, where the diagonal elements are in $\mathbb{R}$ while the off-diagonal ones are in the Jordan algebra J. This suggests that the "three-bit" interpretation of the STU model may be difficult to generalize.

Here, $[D2]^{ij}$ denotes a D2-brane wrapped along the directions ij on T^6, $[D4]_{ij}$ a D4-branes wrapped on all directions *but ij*, $[kk]_i$ a momentum state along direction i, $[kkm]^i$ a Kaluza-Klein 5-monopole localized along the direction i on T^6, $[F1]^i$ a fundamental string winding along direction i, and $[NS5]_i$ a NS5-brane wrapped on all directions but i.

Exercise 12. *Show that in the* $\mathcal{N}=4$ *truncation where only the* $[F1]$, $[kk]$, $[NS5]$, $[kkm]$ *charges are retained,* (99) *reduces to the quartic invariant* (22) *under* $Sl(2)\times SO(6,6)$. *Similarly, in the* $\mathcal{N}=2$ *truncation where only* $[D0]$, $[D2]$, $[D4]$, $[D6]$ *are kept show that one obtains the quartic invariant of a spinor of* $SO^*(12)$, *based on the Jordan algebra* $J_3^{\mathbb{H}}$.

The intermediate (93) also has an interesting interpretation: it is recognized as $1/p^0$ times the entropy $S_{5D}=\pi\sqrt{N(Q)-J^2}$ of a five-dimensional BPS black hole with electric charge and angular momentum

$$Q_A = p^0 q_A + C_{ABC} p^B p^C \tag{101}$$

$$2J_L = \left(p^0\right)^2 q_0 + p^0 p^A q_A + 2N(p) \tag{102}$$

The interpretation of these relations is as follows: When the D6-brane charge p^0 is non-zero, the 4D black hole in Type IIA compactified on $\tilde{Y}$ may be lifted to a 5D black hole in M-theory on $\tilde{Y}\times TN_{p^0}$, where TN denotes the 4-dimensional Euclidean Taub-NUT space with NUT charge p^0; at spatial infinity, this asymptotes to $\mathbb{R}^3\times S^1$, where the circle is taken to be the M-theory direction. Translations along this direction at infinity conjugate to the D0-brane charge q_0 and become $SU(2)$ rotations at the center of TN, where the black hole is assumed to sit. The remaining factors of p^0 are accounted for by taking into account the $\mathbb{R}^4/\mathbb{Z}_{p^0}$ singularity at the origin of TN [72]. The formulae (101) extend this lift to an arbitrary choice of charges in a manifestly duality invariant manner.

Exercise 13. *Using the fact that the degeneracies of five-dimensional black holes on* $K3\times S^1$ *are given by the Fourier coefficients of the elliptic genus of* Hilb$(K3)$, *equal to* $1/\Phi_{10}$, *show that the DVV conjecture* (23) *holds for at least one U-duality orbit of 4-dimensional dyons in type* $II/K3\times T^2$ *with one unit of D6-brane and some amount of D0,D2-brane charge. You might want to seek help from [42].*

4 Topological String Primer

In the previous sections, we were concerned exclusively with low energy supergravity theories, whose Lagrangian contains at most two-derivative terms. This is sufficient in the limit of infinitely large charges, but not for more moderate values, where higher derivative corrections start playing a role. In this section, we give a self-contained introduction to topological string theory, which offers a practical way to compute an infinite series of such corrections. Sections 4.1 and 4.2

draw heavily from [73]. Other valuable reviews of topological string theory include [74, 75, 76, 77, 78].

4.1 Topological Sigma Models

Type II strings compactified on a Kähler manifold X of complex dimension d are described by an $N=(2,2)$ sigma model

$$S = 2t\int d^2z \Big(g_{i\bar{j}}\partial\phi^i\bar{\partial}\phi^{\bar{j}} + g_{i\bar{j}}\bar{\partial}\phi^i\partial\phi^{\bar{j}} + i\psi_-^{\bar{i}} D\psi_-^i g_{i\bar{i}} \\ + i\psi_+^{\bar{i}}\bar{D}\psi_+^i g_{i\bar{i}} + R_{i\bar{i}j\bar{j}}\psi_+^i\psi_+^{\bar{i}}\psi_-^j\psi_-^{\bar{j}} \Big) \tag{103}$$

where ϕ is a map from a two-dimensional genus g Riemann surface Σ to X, $\psi_\pm^i$ is a section of $K_\pm^{1/2}\otimes\phi^*(T^{1,0}X)$, $\psi_\pm^{\bar{i}}$ is a section of $K_\pm^{1/2}\otimes\phi^*(T^{0,1}X)$, and we denoted by K_+ the canonical bundle on Σ (i.e. the bundle of (1,0) forms) and K_- the anti-canonical bundle (of (0,1) forms). The factor of t (the string tension) is to keep track on the dependence on the overall volume of X.

This model is invariant under $N=(2,2)$ superconformal transformations generated with sections $\alpha_\pm$ and $\tilde{\alpha}_\pm$ of $K_\pm^{1/2}$, acting e.g. as

$$\delta\phi^i = i\left(\alpha_-\psi_+^i + \alpha_+\psi_-^i\right)\,,\quad \delta\phi^{\bar{i}} = i\left(\tilde{\alpha}_-\psi_+^{\bar{i}} + \tilde{\alpha}_+\psi_-^{\bar{i}}\right) \tag{104}$$

This implies chirally conserved supercurrents $G^\pm$ of conformal dimension 3/2, which together with T and the current J generate the $\mathcal{N}=2$ superconformal algebra,

$$G^+(z)\,G^-(0) = \frac{2c}{3}\frac{1}{z^2} + \left(\frac{2J}{z^2} + \frac{\partial J + 2T}{z}\right) + \text{reg} \tag{105}$$

$$J(z)J(0) = \frac{c}{3}\frac{1}{z^2} + \text{reg} \tag{106}$$

The current J appearing in the OPE (105) generates a $U(1)$ symmetry, such that $G_\pm$ have charge $Q=\pm 1$ while T and J are neutral. In the (doubly degenerate) Ramond sectors $R_\pm$, the zero-modes of the supercurrents generate a supersymmetry algebra

$$\left(G_0^+\right)^2 = \left(G_0^-\right)^2 = 0\,,\quad \left\{G_0^+, G_0^-\right\} = 2\left(L_0^{R_\pm} - \frac{c}{24}\right) \tag{107}$$

Unitarity forces the right-hand side to be positive on any state. Moreover, the $\mathcal{N}=2$ algebra admits an automorphism known as spectral flow, which relates the NS and R sectors:

$$J_0^{R_\pm} = J_0^{NS} \mp \frac{c}{6}\,,\quad L_0^{R_\pm} = L_0^{NS} \mp \frac{1}{2}J_0^{NS} + \frac{c}{24} \tag{108}$$

The unitary bound $\Delta \geq c/24$ in the R sector therefore implies a bound $\Delta \geq |Q|/2$ after spectral flow. States which saturate this bound have no short distance singularities when brought together and thus form a ring under OPE, known as the *chiral ring* of the $\mathscr{N}=2$ SCFT. Applying the spectral flow twice maps the NS sector back to itself, with $(\Delta, Q) \to (\Delta - Q + \frac{c}{6}, Q \mp \frac{c}{3})$. In particular, the NS ground state is mapped to a state with $(\Delta, q) = (\frac{c}{6}, \mp\frac{c}{3})$ in the chiral ring. For a Calabi-Yau threefold, starting from the identity we thus obtain two R states with $(\Delta, q) = (3/8, \pm 3/2)$ and one NS state with $(\Delta, q) = (3/2, \pm 3)$: these are identified geometrically as the covariantly constant spinor and the holomorphic (3,0) form, respectively.

The spectral flow (108) above can be used to "twist" the $\mathscr{N}=2$ sigma model into a topological sigma model: for this, bosonize the $U(1)$ current $J = i\sqrt{3}\partial H$ so that the spectral flow operator becomes

$$\Sigma_{\pm} = \exp\left(\pm i \frac{\sqrt{3}}{2} H(z)\right) \tag{109}$$

with $(\Delta = 3/8, Q = \pm 3/2)$. The topological twist then amounts to adding a background charge $\pm \int \frac{\sqrt{3}}{2} H\ R^{(2)}$: Its effect is to change the two-dimensional spin L_0 into a linear combination $L_0 \mp \frac{1}{2} J_0$ of the spin and the $U(1)$ charge. Under this operation, choosing the $+$ sign, ψ_+^i becomes a section of $\phi^*(T^{1,0}X)$, i.e. a worldsheet scalar, whereas $\psi_+^{\bar{i}}$ becomes a section of $K_+ \otimes \phi^*(T^{0,1}X)$, i.e. a worldsheet one-form; simultaneously, the supersymmetry parameters α_- and $\tilde{\alpha}_-$ become a scalar and a section of K^{-1}, respectively. Alternatively, we may choose the $-$ sign in (108), where instead ψ_+^i would become a section of $K_+ \otimes \phi^*(T^{1,0}X)$, while $\psi_+^{\bar{i}}$ would turn into a worldsheet scalar. In either case, it is necessary that the canonical bundle K be trivial, in order for the correlation functions to be unaffected by the twist: This is achieved only when computing particular "topological amplitudes" in string theory, which we will discuss in Sect. 5.1.

Since the sigma model (103) has (2,2) superconformal invariance, it is possible to twist both left and right-movers by a spectral flow of either sign. Only the relative choice of sign is important, leading to two very distinct-looking theories, which we discuss in turn:

4.1.1 Topological A-Model

Here, both ψ_+^i and $\psi_-^{\bar{i}}$ are worldsheet scalars and can be combined in a scalar $\chi \in \phi^*(TX)$. On the other hand, ψ_-^i and $\psi_+^{\bar{i}}$ become (0,1) and (1,0) forms $\psi_{\bar{z}}^i$ and $\psi_z^{\bar{i}}$ on the worldsheet. The action is rewritten as

$$S = 2t \int d^2z \left(g_{i\bar{j}} \partial\phi^i \bar{\partial}\phi^{\bar{j}} + g_{i\bar{j}} \bar{\partial}\phi^i \partial\phi^{\bar{j}} + i\psi_z^{\bar{i}} \bar{D}\chi^i g_{i\bar{i}} + i\psi_{\bar{z}}^{\bar{i}} D\chi^{\bar{i}} g_{i\bar{i}} - R_{i\bar{i}j\bar{j}} \psi_{\bar{z}}^i \psi_z^{\bar{i}} \chi^j \chi^{\bar{j}} \right) \tag{110}$$

It allows for a conserved "ghost" charge where $[\phi] = 0, [\chi] = 1, [\psi] = -1$ and is invariant under the scalar nilpotent operator $Q = G_+$,

$$\{Q,\phi^I\} = \chi^I\ , \quad \{Q,\chi^I\} = 0\ , \{Q,\psi^i_{\bar z}\} = i\bar\partial\phi^i - \chi^j\Gamma^i_{jk}\psi^k_{\bar z} \tag{111}$$

The action (110) is in fact Q-exact, up to a total derivative term proportional to the pull-back of the Kähler form $\omega_K = ig_{i\bar j}d\phi^i \wedge d\phi^{\bar j}$, complexified into $J = B + i\omega_K$ by including the coupling to the NS two-form:

$$S = -i\{Q,V\} - t\int_\Sigma \phi^*(J) \tag{112}$$

where V is the "gauge fermion"

$$V = t\int d^2z\, g_{i\bar j}\left(\psi^i_{\bar z}\partial\phi^{\bar j} + \psi^{\bar j}_z\bar\partial\phi^i\right) \tag{113}$$

This makes it clear that the theory is independent of the worldsheet metric, since the energy momentum tensor is Q-exact:

$$T_{\alpha\beta} = \{Q,b_{\alpha\beta}\}\ , \quad b_{\alpha\beta} = \frac{\partial V}{\partial g^{\alpha\beta}} \tag{114}$$

Moreover, the string tension t appears only in the total derivative term so, in a sector with fixed homology class $\int_\Sigma \phi^*(J)$, the semi-classical limit $t \to 0$ is exact. The path integral thus localizes[13] to the moduli space of Q-exact configurations,

$$\partial_{\bar z}\phi^i = 0, \quad \partial_z\phi^{\bar i} = 0, \tag{115}$$

i.e. *holomorphic maps* from Σ to X. Moreover, the local observables of the A-model $O_{\mathscr{W}} = W_{I_1\ldots I_n}\chi^{I_1}\ldots\chi^{I_n}$, where $W_{I_1\ldots I_n}d\phi^{I_1}\ldots d\phi^{I_n}$ is a differential form on X of degree n, are in one-to-one correspondence with the de Rham cohomology of X, since $\{Q,\mathscr{O}_W\} = -\mathscr{O}_{dW}$. Due to an anomaly in the conservation of the ghost charge, correlators of l observables vanish unless

$$\sum_{k=1}^{l} \deg(W_k) = 2d(1-g) + 2\int_\Sigma \phi^*(c_1(X)) \tag{116}$$

The last term vanishes when the Calabi-Yau condition $c_1(X)$ is obeyed. For Calabi-Yau threefolds, at genus 0 the only correlator involves three degree 2 forms,

$$\langle \mathscr{O}_{W_1}\mathscr{O}_{W_2}\mathscr{O}_{W_3}\rangle = \int W_1\wedge W_2\wedge W_3 + \sum_{\beta\in H^{2+}(X)} e^{-t\int_\beta J}\int_\beta W_1\int_\beta W_2\int_\beta W_3 \tag{117}$$

[13] Localization is a general feature of integrals with a fermionic symmetry Q: decompose the space of fields into orbits of Q, parameterized by a Grassman variable θ times its orthogonal complement; since the integrand is independent of θ by assumption, the integral $\int d\theta$ vanishes by the usual rules of Grassmannian integration. This reasoning breaks down at the fixed points of Q, which is the locus to which the integral localizes.

At genus 1, only the vacuum amplitude, known as the elliptic genus of X is non-zero. In Sect. 4.2, we will explain the prescription to construct non-zero amplitudes at any genus, by coupling to topological gravity.

4.1.2 Topological B-Model

The other inequivalent choice consists in twisting $\psi_\pm^{\bar{i}}$ into worldsheet scalars valued in $TX^{0,1}$, while ψ_+^i and $+\psi_-^i$ are (0,1) and (1,0) forms valued in $TX^{1,0}$. Defining $\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}}$, $\theta_i = g_{i\bar{i}}(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}$ and taking $\psi_\pm^i$ as the two components of a one-form ρ^i, the action may be rewritten as

$$S = i\,t\{Q,V\} + t\,W \tag{118}$$

where

$$V = \int_\Sigma d^2z\, g_{i\bar{j}} \left(\rho_z^i \bar{\partial}\phi^{\bar{j}} + \rho_{\bar{z}}^i \bar{\partial}\phi^{\bar{j}}\right) \tag{119}$$

$$W = -\int_\Sigma d^2z \left(\theta_i D\rho^i + \frac{i}{2} R_{i\bar{i}j\bar{j}}\, \rho^i \wedge \rho^j\, \eta^{\bar{i}} \theta_k g^{k\bar{j}}\right) \tag{120}$$

and the nilpotent operator $Q = G_-$ acts as

$$\left\{Q,\phi^i\right\} = 0\,, \quad \left\{Q,\phi^{\bar{i}}\right\} = -\eta^{\bar{i}}\,, \quad \left\{Q,\eta^{\bar{i}}\right\} = \{Q,\theta_i\} = 0\,, \quad \left\{Q,\rho^i\right\} = -id\phi^i \tag{121}$$

Again, the energy-momentum tensor is Q-exact, so that the model is topological. It is also independent of the Kähler structure of X and has a trivial dependence on t, since (apart from contributions from the Q-exact term) t may be reabsorbed by rescaling $\theta \to \theta/t$. The semi-classical limit $t \to \infty$ is therefore again exact, and the path integral localizes on the fixed points of Q, which are now *constant maps*, $d\phi^i = 0$. After localization, the path integral then reduces to an integral over X.

The observables of the B-model are in one-to-one correspondence with degree (p,q) polyvector fields

$$V = V_{\bar{i}_1\ldots\bar{i}_p}^{j_1\ldots j_q}\, d\bar{z}^{\bar{i}_1}\ldots d\bar{z}^{\bar{i}_p}\, \partial_{z^{j_1}}\ldots\partial_{z^{j_q}} \in H^p\left(X, \Lambda^q T^{1,0} X\right) \tag{122}$$

via $d\bar{z}^{\bar{i}} \sim \eta^{\bar{i}}, \partial_{z^j} \sim \theta_j$, since $\{Q,\mathscr{O}_V\} = -\mathscr{O}_{\bar{\partial}V}$. There are now two conserved ghost charges, and the anomaly in the ghost number conservation requires that

$$\sum_{k=1}^{l} p_k = \sum_{k=1}^{l} q_k = d(1-g) \tag{123}$$

For example, at genus 0, the only vanishing correlator on a Calabi-Yau three-fold involves three (1,1) polyvector fields $V_{\bar{j}}^i$. Using the holomorphic (3,0) form, these are related to (2,1) forms $\Omega_{ijl} V_{\bar{k}}^{l}$ parameterizing the complex structure of X. The three-point function is

$$\langle \mathscr{O}_{V_1} \mathscr{O}_{V_2} \mathscr{O}_{V_3} \rangle = \int_X V_{\bar{j}_1}^{i_1} \, V_{\bar{j}_2}^{i_2} \, V_{\bar{j}_3}^{i_3} \Omega_{i_1 i_2 i_3} \, d\bar{z}^{\bar{j}_1} \wedge d\bar{z}^{\bar{j}_2} \wedge d\bar{z}^{\bar{j}_3} \wedge \Omega \tag{124}$$

giving access to the third derivative of the prepotential.

4.2 Topological Strings

Due to the conservation of the ghost number, we have seen that, from the sigma model alone, the only non-vanishing topological correlators are the three-point function on the sphere and the vacuum amplitude on the torus. It turns out that the coupling to topological gravity allows to lift this constraint and define arbitrary n-point amplitudes at any genus.

Recall that in bosonic string theory, genus g amplitudes are obtained by introducing $6g-6$ insertions of the dimension 2 ghost (or, rather, "antighost") b of diffeomorphism invariance, folded with Beltrami differentials $\mu_k \in H^1(\Sigma, T^{1,0}\Sigma)$:

$$F_g = \int_{\mathscr{M}_g} \left\langle \prod_{k=1}^{6g-6} (b, \mu_k) \right\rangle \tag{125}$$

where

$$(b,\mu) = \int_\Sigma d^2 z \left[b_{zz} \mu_{\bar{z}}^{z} + b_{\bar{z}\bar{z}} \, \bar{\mu}_{z}^{\bar{z}} \right] \tag{126}$$

This effectively produces the Weil-Peterson volume element on the moduli space $\mathscr{M}_g$ of complex structures on the genus g Riemann surface Σ (compactified à la Deligne-Mumford). Since b has ghost number -1, this exactly compensates the anomalous background charge.

After the topological twist, which identifies the BRST charge Q with (say) G_+, it is natural to identify b with G_-, in such a way that the energy-momentum tensor is given by $T = \{Q, b\} = \{G_+, G_-\}$. Hence, the genus g vacuum topological amplitude may be written as

$$F_g = \int_{\mathscr{M}_g} \left\langle \prod_{k=1}^{3g-3} (G_-, \mu_k) \, (G_\pm, \bar{\mu}_k) \right\rangle \tag{127}$$

where the upper (resp., lower) sign corresponds to the A-model (resp., B-model). Scattering amplitudes may be obtained by inserting vertex operators with zero ghost number; these may be obtained by "descent" from a ghost number 2 operator $\mathscr{O}^{(0)}$,

$$d\mathscr{O}^{(0)} = \left\{ Q, \mathscr{O}^{(1)} \right\} , \quad d\mathscr{O}^{(1)} = \left\{ Q, \mathscr{O}^{(2)} \right\} \tag{128}$$

Prominent examples of $\mathscr{O}^{(0)}$ are of course $W_{i\bar{i}} \chi^i \chi^{\bar{i}}$ in the A-model and $V_{j}^{\bar{i}} \eta^{\bar{i}} \theta_j$ in the B-model. These describe the deformations of the Kähler and complex structures, respectively. Arbitrary numbers of integrated vertex operators $\int d^2z\, \mathscr{O}^{(2)}$ can then be inserted in (127) without spoiling the conservation of ghost charge number.

Weighting the contributions of different genera by powers of the "topological string coupling" λ, namely,

$$F_{\rm top} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g \tag{129}$$

we obtain obtain a perturbative definition of the A and B-model topological strings. Since the worldsheet is topological, the target space theory has only a finite number of fields, so it is really more a field theory than a string theory. In fact, the tree-level scattering amplitudes can be reproduced by a simple action X, known as "holomorphic Chern-Simons" in the A-model and "Kodaira-Spencer" in the B-model; these describe the fluctuations of Kähler and complex structures, respectively. We refer the reader to [79] for an extensive discussion of these theories.

4.3 Gromov-Witten, Gopakumar-Vafa and Donaldson-Thomas Invariants

We now concentrate on the topological vacuum amplitude (129) of the A-model on a Calabi-Yau threefold X. Up to holomorphic anomalies that we discuss in the next section, $F_{\rm top}$ can be viewed as a function of the complexified Kähler moduli $t^A = \int_{\gamma^A} J$. In the large volume limit (or more generally, near a point of maximal unipotent monodromy), it has an asymptotic expansion

$$F_{\rm top} = -i\frac{(2\pi)^3}{6\lambda^2} C_{ABC} t^A t^B t^C - \frac{i\pi}{12} c_{2A} t^A + F_{GW} \tag{130}$$

where C_{ABC} are the triple intersection numbers of the 4-cycles γ_A dual to γ^A, and $c_{2A} = \int_{\gamma_A} c_2(T^{(1,0)}X)$ are the second Chern classes of these 4-cycles. The first two terms in (130) are perturbative in α', while F_{GW} contains the effect of worldsheet instantons at arbitrary genus,

$$F_{GW} = \sum_{g\geq 0} \sum_{\beta \in H_2^+(X)} N_{g,\beta}\, e^{2\pi i \beta_A t^A} \lambda^{2g-2} \tag{131}$$

where the sum runs over effective curves $\beta = \beta_A \gamma^A$ with $\beta_A \geq 0$, and N_g^β are (conjecturally) rational numbers known as the Gromov-Witten (GW) invariants of X. It is possible to re-organize the sum in (131) into

$$F_{GW} = \sum_{g\geq 0} \sum_{\beta \in H_2^+(X)} \sum_{d\geq 1} n_{g,\beta}\, \frac{1}{d} \left[2\sin\left(\frac{d\lambda}{2}\right)\right]^{2g-2} e^{2\pi i d \beta_A t^A} \tag{132}$$

The coefficients $n_{g,\beta}$ are known as the Gopakumar-Vafa (GV) invariants and are conjectured to always be integer: indeed, one may show that the contribution of a fixed β_A in (132) arises from the one-loop contribution of a M2-brane wrapping the isolated holomorphic curve $\beta^A\gamma_A$ in X [80, 81]. The GV invariants can be related to the GW invariants by expanding (132) at small λ and matching on to (131), e.g. at leading order λ^{-2},

$$N_{0,\beta} = \sum_{d|\beta_A} d^3\, n_{0,\beta^A/d} \tag{133}$$

which incorporates the effect of multiple coverings for an isolated genus 0 curve.

It should be noted that the sum in (131) or (132) includes the term $\beta = 0$, which corresponds to degenerate worldsheet instantons. It turns out that the only non-vanishing GV invariant at genus 0 is $n_{0,0} = -\frac{1}{2}\chi(X)$, hence

$$F_{\rm deg} = -\frac{1}{2}\chi(X) \sum_{d\geq 1} \frac{1}{d}\left[2\sin\left(\frac{d\lambda}{2}\right)\right]^2 \equiv -\frac{1}{2}\chi(X)\, f(\lambda) \tag{134}$$

The function $f(\lambda)$, known as the Mac-Mahon function, may be formally manipulated into

$$f(\lambda) = -\sum_{d\geq 1} \frac{e^{id\lambda}}{d(1-e^{id\lambda})^2} = -\sum_{d=1}^{\infty}\sum_{n=1}^{\infty} \frac{n\, q^{nd}}{d} = \sum_{n=1}^{\infty} n\, \log(1-q^n) \tag{135}$$

where $q = e^{i\lambda}$. The last expression converges in the upper half plane $\mathrm{Im}(\lambda) > 0$, and may be taken as the definition of the Mac-Mahon function, suitable in the large coupling limit $\lambda \to i\infty$.

Exercise 14. *Check that the coefficient of q^N in the Taylor expansion of* $\exp(-f)$ *counts the number of three-dimensional Young tableaux with N boxes.*

In order to analyze its contributions at weak coupling $t = -i\lambda \to 0$, let us compute its Mellin transform[14]

$$M(s) = \int_0^\infty \frac{dt}{t^{1-s}} f(t) = -\int_0^\infty \frac{dt}{t^{1-s}} \sum_{d=1}^{\infty}\sum_{n=1}^{\infty} \frac{n}{d} e^{-ndt} \tag{136}$$

Exchanging the integral and sums, the result is simply expressed in terms of Euler Γ and Riemann ζ functions,

$$-\sum_{d=1}^{\infty}\sum_{n=1}^{\infty} \frac{n}{d}(nd)^{-s}\Gamma(s) = -\zeta(s-1)\zeta(s+1)\Gamma(s) \tag{137}$$

The function $f(t)$ itself may be obtained conversely by

[14] The following argument, due to S. Miller (private communication), considerably streamlines the computation in [6].

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} M(s)\, t^{-s} \tag{138}$$

where the contour is chosen to lie to the right of any pole of $M(s)$. Moving the contour to the left and crossing the poles generate the Laurent series expansion of $f(t)$.

To perform this computation, recall that $\Gamma(s)$ has simple poles at $s = -n, n = 0, 1, \dots$ with residue $(-1)^n/n!$. Moreover, $\zeta(s)$ has a simple pole at $s = 1$, and "trivial" zeros at $s = -2, -4, -6, \dots$. The trivial zeros of $\zeta(s-1)$ and $\zeta(s+1)$ cancel the poles of $\Gamma(s)$ at odd negative integer, leaving only the simple poles at even strictly negative integer, a double pole at $s = 0$ and a single pole at $s = 2$. Altogether, returning to the variable $\lambda = it$,we obtain the Laurent series expansion

$$f(\lambda) = \frac{\zeta(3)}{\lambda^2} + \frac{1}{12}\log(i\lambda) - \zeta'(1) + \sum_{g=2}^{\infty} \frac{B_{2g}B_{2g-2}\lambda^{2g-2}}{(2g-2)!(2g-2)(2g)} \tag{139}$$

where we further used the relation $\zeta(3-2g) = -B_{2g-2}/(2g-2)$ $(g \geq 2)$ between the values of ζ and Bernoulli numbers.

The leading term, proportional to $\zeta(3)$, leads to a constant shift $-1/2\chi(X)\zeta(3)$ in the tree-level prepotential, and can be traced back to the tree-level R^4 term in the 10-dimensional effective action, reduced along X [82, 83, 84]. The terms with $g \geq 2$ were first computed using heterotic/type II duality [85] and impressively agree with an independent computation of the integral over the moduli space $\mathscr{M}_g$ [86],

$$\int_{\mathscr{M}_g} c_{g-1}^3 = -\frac{B_{2g}B_{2g-2}\lambda^{2g-2}}{(2g-2)!(2g-2)(2g)} \tag{140}$$

The logarithmic correction in (139) originates from the double pole of $M(s)$ at $s = 0$ and has no simple interpretation yet. It is nevertheless forced, if one accepts that the correct non-perturbative completion of the degenerate instanton series is the MacMahon function [5, 6].

For completeness, let us finally mention the relation to a third type of topological invariants, known as Donaldson-Thomas invariants $n_{DT}(q_A, m)$ [87]: these count "ideal sheaves" on X, which can be understood physically as bound states of m D0-branes, q_A D2-branes wrapped on $q^A\gamma_A \in H_2(\mathbb{Z})$ and a single D6-brane. S-duality implies [88, 89] that the partition function of Donaldson-Thomas invariants is related to the partition function of Gromov-Witten invariants by [90, 91]

$$\sum_{q^A \in H_2(\mathbb{Z}), m \in \mathbb{Z}} n_{DT}(q_A, m)\, e^{it_A q^A} q^m = \exp\left[F_{GW}(t, \lambda) - \frac{\chi}{2} f(\lambda)\right] \tag{141}$$

where $q = -e^{i\lambda}$. Such a relation may be understood from the fact that a curve may be represented either by a set of a equations (the Donaldson-Thomas side) or by an explicit parameterization (the Gromov-Witten side). This conjecture has been recently proven for any toric three-fold X [92].

4.4 Holomorphic Anomalies and the Wave Function Property

In the previous subsection, we assumed that the topological amplitude was a function of the holomorphic moduli t^i only. This is naively warranted by the fact that the variation of the anti-holomorphic moduli $\bar{t}^{\bar{i}}$ results in the insertion of an (integrated) Q-exact operator, $\phi_{\bar{i}} = \{G^+, [\bar{G}^+, \bar{\phi}_{\bar{i}}]\}$. By the same naive reasoning, one would expect that the n-point functions $C^{(g)}_{i_1\dots i_n}$ be independent of $\bar{t}$, and equal to the n-th order derivative of the vacuum amplitude F_g with respect to $t^{i_1},\dots t^{i_n}$. Both of these expectations turn out to be wrong, due to boundary contributions in the integral over the moduli space of genus g Riemann surfaces. By analyzing these contributions carefully, Berschadsky, Cecotti, Ooguri, and Vafa [79] (BCOV) have shown that the $\bar{t}^i$ derivative of F_g is related to $F_{h<g}$ at lower genera via[15]

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} e^{2\mathscr{K}} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{h=1}^{g-1} (D_j F_h)(D_k F_{g-h}) \right) \tag{142}$$

where $D_i F_g = (\partial_i - (2-2g)\partial_i \mathscr{K}) F_g$, as appropriate for a section of $\mathscr{L}^{2-2g}$, where $\mathscr{L}$ is the Hodge bundle defined below (29). In (142), the first term on the right-hand side originates from the boundary of $\mathscr{M}_g$ where one non-contractible handle of Σ is pinched, whereas the second term corresponds to the limit where a homologically trivial cycle vanishes, disconnecting Σ into two Riemann surfaces with genus h and $g-h$. A similar identity can be derived for n-point functions. Moreover, the latter are indeed obtained from the vacuum amplitude by derivation with respect to t^i, provided one uses a covariant derivative taking into account the Levi-Civita and Kähler connections:

$$C^{(g)}_{i_1\dots i_n} = \begin{cases} D_{i_1}\dots D_{i_n} F_g & \text{for } g \geq 1, n \geq 1 \\ D_{i_1}\dots D_{i_{n-3}} C_{i_{n-1} i_{n-1} i_n} & \text{for } g = 0, n \geq 3 \\ 0 & \text{for } 2g-2+n \leq 0 \end{cases} \tag{143}$$

where C_{ijk} is the tree-level three-point function. The resulting identities may be summarized by defining the "topological wave-function"

$$\Psi_{\mathrm{BCOV}} = \lambda^{\frac{\chi}{24}-1} \exp\left[\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1\dots i_n} x^{i_1} \dots x^{i_n} \right] \tag{144}$$

Note that Ψ_{BCOV} does *not* incorporate the genus 1 vacuum amplitude. In terms of this object, the identities (142) (or rather their generalization to n-point functions) and (143) are summarized by the two equations

$$\partial_{\bar{t}^i} = \frac{\lambda^2}{2} e^{2\mathscr{K}} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - g_{\bar{i}j} x^j \left(\lambda \frac{\partial}{\partial \lambda} + x^k \frac{\partial}{\partial x^k} \right) \tag{145}$$

[15] When $g = 1$, the holomorphic equation becomes second order and can be read off from (145) below.

$$\partial_{t^i} = \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} - \partial_i \mathscr{K} \left(\frac{\chi}{24} - 1 - \lambda \frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial x^i} - \partial_i F_1 - \frac{1}{2\lambda^2} C_{ijk} x^j x^k \tag{146}$$

By rescaling $x^i \to \lambda x^i, \Psi \to e^{f_1(t)} \Psi_V$ where $f_1(t)$ is the holomorphic function in the general solution

$$F_1 = -\frac{1}{2} \log |g| + \left(\frac{n_V + 1}{2} - \frac{\chi}{24} + 1 \right) \mathscr{K} + f_1(t) + \bar{f}_1(\bar{t}) \tag{147}$$

of the holomorphic anomaly equation for F_1, E. Verlinde [93] was able to recast (145), (146) in a form involving only special geometry data,

$$\partial_{\bar{t}^i} = \frac{1}{2} e^{2\mathscr{K}} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} + g_{\bar{i}j} x^j \frac{\partial}{\partial \lambda^{-1}} \tag{148}$$

$$\nabla_i - \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} = \frac{1}{2} \partial_{t^i} \log |g| + \frac{1}{\lambda} \frac{\partial}{\partial x^i} - \frac{1}{2} e^{-2\mathscr{K}} C_{ijk} x^j x^k \tag{149}$$

where

$$\nabla_i = \partial_i + \partial_i \mathscr{K} \left(x^k \frac{\partial}{\partial x^k} - \lambda \frac{\partial}{\partial \lambda} + \frac{n_V + 1}{2} \right) \tag{150}$$

Here, $|g| = \det(g_{i\bar{j}})$. The implications of these equations were understood in [94] and further clarified in [9, 93, 95]: $\Psi(t, \bar{t}; x, \lambda)$ should be thought of as a single state $|\Psi\rangle$ in a Hilbert space, expressed on a $(t, \bar{t})$-dependent basis of coherent states,

$$\Psi_V(t, \bar{t}; x, \lambda) =_{(t,\bar{t})} \langle x^i, \lambda | \Psi \rangle \tag{151}$$

This is most easily explained in the B-model, where (x, λ^{-1}) and their complex conjugate can be viewed as the coordinates of a 3-form $\gamma \in H^3(X, \mathbb{R})$ on the Hodge decomposition

$$\gamma = \lambda^{-1} \Omega + x^i D_i \Omega + x^{\bar{i}} D_{\bar{i}} \bar{\Omega} + \bar{\lambda}^{-1} \bar{\Omega} \tag{152}$$

The space $H^3(X, \mathbb{R})$ admits a symplectic structure

$$\omega = i\, e^{-\mathscr{K}} \left(g_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}} - d\lambda^{-1} \wedge d\bar{\lambda}^{-1} \right) \tag{153}$$

inherited from the anti-symmetric pairing $(\alpha, \beta) = \int_X \alpha \wedge \beta$, which leads to the Poisson brackets between the coordinates

$$\left\{ \lambda^{-1}, \bar{\lambda}^{-1} \right\} = i\, e^{\mathscr{K}} \ , \quad \left\{ x^i, \bar{x}^{\bar{j}} \right\} = -i g^{i\bar{j}} \tag{154}$$

The phase $H^3(X, \mathbb{R})$ may be quantized by considering functions (or rather half-densities, to account for the zero-point energy) of (λ^{-1}, x^i) and representing $\bar{\lambda}^{-1}$ and $\bar{x}^{\bar{i}}$ as derivative operators,

$$\bar{\lambda}^{-1} = -e^{\mathscr{K}} \frac{\partial}{\partial \lambda^{-1}} \ , \quad \bar{x}^{\bar{i}} = e^{\mathscr{K}} g^{\bar{i}j} \frac{\partial}{\partial x^j} \tag{155}$$

The resulting wave function $\Psi(t,\bar{t};\lambda,x)$ carries a dependence on the "background" variables $(t,\bar{t})$ since the decomposition (152) does depend on these variables via Ω. A variation of t and $\bar{t}$ generically mixes (λ^{-1},x) with their canonical conjugate and so may be compensated by an infinitesimal Bogolioubov transformation, reflected in (148), (149). In fact, we can check that these two equations are hermitean conjugate under the inner product

$$\begin{aligned}\langle\Psi'|\Psi\rangle = \int dx^i d\bar{x}^{\bar{i}} d\lambda^{-1} d\bar{\lambda}^{-1} |g|\, e^{-\frac{n_V+1}{2}\mathcal{K}} \\ \exp\left(-e^{-\mathcal{K}} x^i g_{i\bar{j}} \bar{x}^j + e^{-\mathcal{K}} \lambda^{-1}\bar{\lambda}^{-1}\right) \Psi'^*\left(t,\bar{t};\bar{x},\bar{\lambda}\right) \Psi(t,\bar{t};x,\lambda)\end{aligned} \tag{156}$$

which is the natural inner product arising in Kähler quantization. In contrast to Ψ and Ψ' separately, the inner product is background independent (and, in fact, a pure number), by virtue of the anomaly equations.

Exercise 15. *Show that in the harmonic oscillator Hilbert space, the wave functions in the real and oscillator polarizations are related by (abusing notation)*

$$f(q) = \int da^\dagger\; e^{ia^\dagger q\sqrt{2}+q^2/2-(a^\dagger)^2/2} f(a^\dagger) = \int da\; e^{-iaq\sqrt{2}-q^2/2+a^2/2} f(a) \tag{157}$$

Conclude that the inner product in oscillator basis is given by

$$\int dq\; f^*(q) g(q) = \int da da^\dagger\; e^{-aa^\dagger} f^*(a) g(a^\dagger) \tag{158}$$

This observation suggests that there exists a different background independent polarization obtained by choosing a real symplectic basis γ^I, γ_I of three-cycles in $H_3(X,\mathbb{Z})$, and expanding

$$\gamma = p^I \gamma_I + q_I \gamma^I \tag{159}$$

The symplectic form is now just $\omega = dq_I \wedge dp^I$, so $H_3(X,\mathbb{R})$ can be quantized by considering functions of p^I and representing q_I as $i\partial/\partial p^I$; equivalently, one may introduce a set of coherent states $|p^I\rangle$ and define the wave function in the "real" polarization,

$$\Psi_{\mathbb{R}}(p^I) = \langle p^I|\Psi\rangle. \tag{160}$$

This is related to the wave function in the Kähler polarization by a finite Bogolioubov transformation[16]

$$\Psi_{\mathbb{R}}(p^I) = \int dx^i\; d\lambda\; \langle p^I|x^i,\lambda\rangle\; \Psi_V\,(t,\bar{t};\lambda,x) \tag{161}$$

The overlap of coherent states $\langle p^I|x^i,\ \lambda\rangle$ is a solution of the equations hermitian-conjugate to (148), (149) [9, 93],

[16] A precursor of this formula was already found in [79], although not recognized as such.

$$\langle p^I | x^i, \lambda \rangle = e^{-(n_V+1)\mathcal{K}/2} \sqrt{\det g_{i\bar{j}}} \exp \Big[-\frac{1}{2} p^I \bar{\tau}_{IJ} p^J + 2ip^I [\mathrm{Im}\tau]_{IJ} \left(\lambda^{-1} X^I + e^{-\mathcal{K}/2} x^i f_i^I \right) + i \left(\lambda^{-2} X^I [\mathrm{Im}\tau]_{IJ} X^J + 2\lambda^{-1} e^{-\mathcal{K}/2} x^i f_i^I [\mathrm{Im}\tau]_{IJ} X^J + e^{-\mathcal{K}} x^i f_i^I [\mathrm{Im}\tau]_{IJ} f_j^J x^j \right) \Big] \tag{162}$$

While the topological wave function in the real polarization has the great merit of being background independent, it is nevertheless not canonical, since it depends on a choice of symplectic basis. As usual in quantum mechanics, changes of symplectic basis are implemented by the metaplectic representation of $Sp(2n_V+2)$ (or rather, of its metaplectic cover). In particular, upon exchanging A and B cycles, $\Psi_{\mathbb{R}}(p^I)$ is turned into its Fourier transform, which is the quantum analogue of the classical property discussed in Exercise 4 on page 14.

For completeness, let us mention that there exists a different "holomorphic" polarization, intermediate between the Kähler and real polarizations, where the topological amplitude is a purely holomorphic function of the background moduli t^i, satisfying a heat-type equation analogous to the Jacobi theta series [9]. Moreover, for "very special supergravities", the holomorphic anomaly equations can be traced to operator identities in the "minimal" representation of the three-dimensional duality group QConf(J); this is analogous to the case of the Jacobi theta series, where the Siegel modular group $Sp(4,\mathbb{Z})$ plays the role of QConf(J). This hints at the existence of a one-parameter generalization of the topological string amplitude, which we return to in Sect. 7.5.3.

5 Higher Derivative Corrections and Topological Strings

In this section, we return to the realm of physical string theory and explain how a special class of higher derivative terms in the low-energy effective action can be reduced to a topological string computation. We then discuss how these terms affect the Bekenstein-Hawking entropy of black holes and formulate the Ooguri-Strominger-Vafa conjecture, which purportedly relates the topological amplitude to the microscopic degeneracies.

5.1 Gravitational F-Terms and Topological Strings

In general, higher derivative and higher genus corrections in string theory are very hard to compute: The integration measure on supermoduli space is ill-understood beyond genus 2 (see [96] for the state of the art at genus 2), and the current computation schemes (with the exception of the pure spinor superstring, see e.g. [97]) are non-manifestly supersymmetric, requiring to evaluate many different scattering amplitudes at a given order in momenta.

Fortunately, $\mathscr{N}=2$ supergravity coupled to vector multiplets has an off-shell superspace description, which greatly reduces the number of diagrams to be computed, and also provides a special family of "F-term" interactions, which can be efficiently computed. The most convenient formulation starts from $\mathscr{N}=2$ conformal supergravity and fixes the conformal gauge so as to reduce to Poincaré supergravity (see [50] for an extensive review of this approach). The basic objects are the Weyl and matter chiral superfields,

$$W_{\mu\nu}(x,\theta) = T_{\mu\nu} - \frac{1}{2} R_{\mu\nu\rho\sigma} \epsilon_{\alpha\beta} \theta^\alpha \sigma_{\lambda\rho} \theta^\beta + \dots \tag{163}$$

$$\Phi^I(x,\theta) = X^I + \frac{1}{2} \mathscr{F}^I_{\mu\nu} \epsilon_{\alpha\beta} \theta^\alpha \sigma^{\mu\nu} \theta^\beta + \dots \tag{164}$$

where $\alpha,\beta = 1,2$. $T_{\mu\nu}$ is an auxiliary anti-selfdual tensor, identified by the (tree-level) equations of motion as the graviphoton (40). From W, one may construct the scalar chiral superfield

$$W^2(x,\theta) = T_{\mu\nu}T^{\mu\nu} - 2\epsilon_{ij}\theta^i \sigma^{\mu\nu}\theta^j R_{\mu\nu\lambda\rho} T^{\lambda\rho} - (\theta^i)^2(\theta^j)^2 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \dots \tag{165}$$

where the anti-self dual parts of R and T are understood. Starting with any holomorphic, homogeneous of degree two function $F(\Phi^I,\ W^2)$, regular at $W^2=0$,

$$F(\Phi^I, W^2) \equiv \sum_{g=0}^{\infty} F_g(\Phi^I) W^{2g} \tag{166}$$

(where F_g is homogeneous of degree $2-2g$) one may construct the chiral integral

$$\int d^4\theta d^4x\, F(\Phi, W^2) = S_{\text{tree}} + \int \sum_{g=1}^{\infty} F_g(X^I) \left(g\, R^2 T^{2g-2} + 2g(g-1)(RT)^2 T^{2g-4} \right) + \dots \tag{167}$$

which reproduces the tree-level $\mathscr{N}=2$ supergravity action based on the prepotential F_0, plus an infinite sum of higher derivative "F-term" gravitational interactions (plus non-displayed terms). $F(\Phi^I,\ W^2)$ is known as the generalized prepotential.

In order to compute the coefficients $F_g(X^I)$, one should compute the scattering amplitude of 2 gravitons and $2g-2$ graviphotons in type II (A or B) string theory at leading order in momenta compactified on a Calabi-Yau threefold X. This problem was studied in [98], where it was shown (as anticipated in [79]) that it reduces to a computation in topological string theory. We now briefly review the argument.

The graviphoton originates from the Ramond-Ramond sector; taking into account the peculiar couplings of RR states to the dilaton, F_g is identified as a genus g amplitude[17]. Perturbative contributions from a different loop order or non-perturbative ones are forbidden, since the type II dilaton is an hypermultiplet. The graviton vertex operator (in the 0 superghost picture) is

[17] When X is K3-fibered, and in the limit of a large base, one can obtain the generalized prepotential from a one-loop heterotic computation [85, 99].

$$V_g^{(0)} = h_{\mu\nu}(\partial X^\mu + ip\cdot\psi\,\psi^\mu)(\bar\partial X^\mu + ip\cdot\tilde\psi\,\tilde\psi^\mu)e^{ipX} \tag{168}$$

The vertex operator of the graviphoton (in the $-1/2$ superghost picture) is

$$V_T^{(-1/2)} = \epsilon_\mu p_\nu e^{-(\phi+\tilde\phi)/2}\left(S\sigma_{\mu\nu}\tilde S\,\Sigma_+\tilde\Sigma_\mp + cc\right)e^{ipX} \tag{169}$$

where $S,\tilde S$ are spin fields in the 4 non-compact dimensions, and $\Sigma_\pm$ is the spectral flow operator (109) in the $N=(2,2)$ SCFT. The insertion of $2g-2$ graviphotons induces a background charge $\int \frac{\sqrt{3}}{2}H\,R^{(2)}$, which induces the topological twist $L_0 \to L_0 - \frac{1}{2}J$. The same process takes place in the SCFT describing the 4 non-compact directions. As a result, the bosonic and fermionic fluctuation determinants cancel. Moreover, choosing the polarizations of the graviton and graviphotons to be anti-self-dual, only the $\psi\psi\tilde\psi\tilde\psi$ terms in (168) contribute after summing over spin structures and cancel against the contractions of the spin fields $S\tilde S$.

Now we turn to the cancelation of the superghost charge: The integration over supermoduli brings down $2g-2$ powers of the picture-changing operator $e^\phi T_F \times cc$, where $T_F = G_+ + G_-$ is the supercurrent. In order to cancel the superghost background charge $2g-2$, it is therefore necessary to transform $g-1$ of the $2g-2$ graviphoton vertex operators in the $+1/2$ picture. In total, we thus have $3g-3$ insertions of T_F. By conservation of the $U(1)$ charge, it turns out that only the G_- and $\tilde G_\pm$ parts of T_F and $\tilde T_F$ contribute. Finally, we reach

$$A_g = (g!)^2\int_{\mathscr{M}_g}\left\langle \prod_{a=1}^{3g-3}(\mu_a G_-)\left(\tilde\mu_a\tilde G_\pm\right)\right\rangle = (g!)^2 F_g \tag{170}$$

where the upper (lower) sign corresponds to type IIB (resp. IIA). We conclude that the generalized prepotential $F_g(X)$ in type IIA (B) string theory compactified on X is equal to the all genus vacuum amplitude (129) of the A (resp. B)-model topological string. The precise identification of the variables is

$$F_{\rm top} = \frac{i\pi}{2}F_{SUGRA}\ , \quad t^A = \frac{X^A}{X^0}\ , \quad \lambda = \frac{\pi}{4}\frac{W}{X^0} \tag{171}$$

To be more precise, the vacuum topological amplitude $F_g(t,\bar t)$, computes the physical R^2T^{2g-2} coupling; it differs from the the holomorphic "Wilsonian" coupling $F_g(X)$ appearing in (167) due to the contributions of massless particles. It is often assumed that these contributions are removed by taking $\bar t \to \infty$ keeping t fixed; it would be interesting to determine whether this is indeed equivalent to going to using the real polarized topological wave function (161).

For completeness and later reference, let us mention that, by a similar reasoning, the topological B-model (resp. A) in type IIA (resp. B) computes higher-derivative interactions between the hypermultiplets, of the form [98]

$$\tilde S = \int d^4x \sum_{g=1}^\infty \tilde F_g(X)\left[g(\partial\partial S)^2(\partial Z)^{2g-2} + 2g(g-1)(\partial\partial S\partial Z)^2(\partial Z)^{2g-4}\right] \tag{172}$$

where (S,Z) describes the universal hypermultiplet. It is also an interesting open problem to construct an off-shell superfield formalism which would describe all these interactions at once as F-terms.

5.2 Bekenstein-Hawking-Wald Entropy

In general, higher derivative corrections affect the macroscopic entropy of black holes in two ways:

(i) they affect the actual solution, and in particular the relation between the horizon geometry and the data measured at infinity;
(ii) by modifying the stress-energy tensor, they change the relation between geometry and entropy.
Moreover, since subleading contributions to the statistical entropy are non-universal, comparison with the microscopic result requires
(iii) specifying the statistical ensemble implicit in the low-energy field theory.

As far as (i) is concerned, and provided we restrict to BPS black holes, the fact that the generalized $\mathcal{N}=2$ supergravity has an off-shell description simplifies the computation drastically: The supersymmetry transformation rules are the same as at tree-level; Cardoso, de Wit, and Mohaupt [100, 101, 102, 103] (CdWM) have shown that the horizon geometry is still $AdS_2\times S^2$, while the value of the moduli is governed by the a generalization of the stabilization (69),

$$\mathrm{Re}(Y^I)=p^I\ ,\quad \mathrm{Re}(G_I)=q_I\ ,\quad W^2=2^8 \tag{173}$$

where G_I is now the derivative of the generalized prepotential, $G_I=\partial F(Y,W^2)/\partial Y^I$.

As far as (ii) is concerned, Wald [104] has given a general prescription for obtaining an entropy functional that satisfies the first law[18] of thermodynamics, in the context of a Lagrangian $\mathscr{L}(R)$ with a general dependence on the Riemann tensor:

$$S_{BHW}=2\pi\int_\Sigma \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\,\sqrt{h}\,d\Omega \tag{174}$$

where h is the induced metric on the horizon Σ, and $\epsilon^{\mu\nu}$ is the binormal.

Exercise 16. *Show that for* $\mathscr{L}=-\frac{1}{16\pi G}R$, (174) *reduces to the usual Bekenstein-Hawking area law.*

While the $\mathcal{N}=2$ corrected Lagrangian does not have such a simple form, CdWM adapted Wald's construction and found a simple result generalizing (70)

$$S_{BHW}=\frac{i\pi}{4}\left(\bar{Y}^I G_I-Y^I\bar{G}_I\right)-\frac{\pi}{2}\mathrm{Im}\left[W\partial_W F\right] \tag{175}$$

[18] The validity of the zero-th and second law was discussed in [105, 106].

where the right-hand side should be evaluated at the attractor point (173).

It should be emphasized that this result takes into account the contributions of the F-terms only; at a given order in momenta, there surely are other "D-terms" interactions which would contribute to the thermodynamical entropy. The results below suggest that such contributions should cancel for BPS black holes: a beautiful proof has been given in [107] but it assumes that the black hole can be lifted to 5 dimensions.

5.3 The Ooguri-Strominger-Vafa Conjecture

As noticed in [4], using the homogeneity relation $Y^I G_I + W\partial_W F = 2F$, it is possible to perform the same manipulation as in (72) and rewrite the entropy (175) as a Legendre transform

$$S_{BHW} = \frac{i\pi}{4}\left[(Y^I - 2i\phi^I)G_I - (\bar{Y}^I + 2i\phi^I)\bar{G}_I\right] + \frac{i\pi}{4}\left[W\partial_W F - \bar{W}\partial_{\bar{W}}\bar{F}\right] \quad (176)$$

$$= \frac{i\pi}{2}(F - \bar{F}) + \frac{\pi}{2}\phi^I(G_I + \bar{G}_I) \quad (177)$$

$$= \mathscr{F}(p^I, \phi^I) + \pi\phi^I q_I \quad (178)$$

of the "topological free energy" $\mathscr{F}(p^I, \phi^I)$, which now incorporates the infinite series of higher derivative F-term corrections,

$$\mathscr{F}(p^I, \phi^I) = -\pi \,\mathrm{Im}\left[F(Y^I = p^I + i\phi^I; W^2 = 2^8)\right] \quad (179)$$

In fact, there are now general arguments [107, 108] to the effect that the Bekenstein-Hawking-Wald entropy is equal the Legendre transform of the Lagrangian evaluated on the near-horizon geometry; in the case of $\mathscr{N} = 2$ supergravity, the equality of this Lagrangian with the topological free energy $\mathscr{F}(p, \phi)$ was checked recently in [28].

As argued by OSV, the simplicity of (178) strongly suggests that the thermodynamical ensemble implicit in the BHW entropy is a "mixed" ensemble, where magnetic charges are treated micro-canonically but electric charges are treated canonically; the thermodynamical relation (178) should then perhaps be viewed as an approximation of an exact relation between two different statistical ensembles

$$\sum_{q_I \in \Lambda_{el}} \Omega(p^I, q_I) e^{-\pi\phi^I q_I} \stackrel{?}{=} e^{\mathscr{F}(p^I, \phi^I)} \quad (180)$$

where $\Omega(p^I,\, q_I)$ are the "microcanonical" degeneracies of states with fixed charges (p^I, q_I), and the sum runs over the lattice Λ_{el} of electric charges. Making use of (179), the right-hand side may be rewritten as

$$\sum_{q_I \in \Lambda_{el}} \Omega\left(p^I, q_I\right) e^{-\pi\phi^I q_I} \stackrel{?}{=} \left|\Psi_{\mathrm{top}}\left(p^I + i\phi^I, 2^8\right)\right|^2 \quad (181)$$

or, conversely,

$$\Omega(p^I, q_I) \stackrel{?}{=} \int d\phi^I \, |\Psi_{\rm top}(p^I + i\phi^I, 2^8)|^2 e^{\pi\phi^I q_I} \tag{182}$$

It should be stressed that going from the "OSV fact" (75) to the OSV conjecture (181) involves a considerable leap of faith which should not be taken lightly.

In its strongest form, the conjecture provides a way to compute the exact microscopic degeneracies $\Omega(p^I,\ q_I)$ from the topological string amplitude $F(X,W^2)$. However, this would most likely require extending the definition of $F(X,W^2)$ to include non-perturbative contributions in W. Conversely, one may hope to understand the non-perturbative completion of the topological string from a detailed knowledge of black hole degeneracies. The weaker, more concrete form of the OSV conjecture states that the relation (182) should hold asymptotically to all orders in inverse charges.

The conjecture calls for some immediate remarks:

- While the formula (182) at first sight seems to treat electric and magnetic charges differently, it is nevertheless invariant under electric-magnetic duality, provided the topological amplitude $\Psi_{\rm top}$ transforms in the metaplectic representation of the symplectic group (see Exercise 18 on page 57 below). Thus, $\Psi_{\rm top}$ should be understood as the topological wave function $\Psi_{\mathbb{R}}(p^I)$ in the real polarization [93], which may be different from the $\bar{t} \to \infty$ limit, as stressed below (171).
- Upon analytically continuing $\phi^I = i\chi^I$, the right-hand side of (182) defines the Wigner function associated to the quantum state $\Psi_{\rm top}$ (we shall return to this observation in Sect. 7). As is well known in quantum mechanics, it is not definite positive, so if the strong conjecture is to hold, $\Omega(p,q)$ should probably refer to an index rather than to an absolute degeneracy of states. This fits well with the fact that $\Psi_{\rm top}$ contains only information about F-term interactions, which is probably insufficient to encode the absolute degeneracies.
- Due to charge quantization, the left-hand side of (181) is formally periodic under imaginary shifts $\phi^I \to \phi^I + 2ik^I,\ k^I \in \mathbb{Z}$, which is not the case of the right-hand side $|\Psi_{\rm top}|^2$. This can be repaired by replacing (181) by

$$\sum_{q_I \in \Lambda_{el}} \Omega(p^I, q_I) e^{-\pi\phi^I q_I} \stackrel{?}{=} \sum_{k^I \in \Lambda^*_{el}} \Psi^*\left(p^I - 2k^I - i\phi^I\right)\, \Psi\left(p^I + 2k^I + i\phi^I\right) \tag{183}$$

  without affecting the converse statement (182). The right-hand side of this equation is reminiscent of a theta series. Similar averaging have indeed been found to occur in some non-compact models [109, 110]. Note however that this averaging renders the prospect of recovering the non-perturbative generalization of $\Psi_{\rm top}$ from $\Omega(p^I, q_I)$ more uncertain.
- The sum on the left-hand side of (181) does not appear to converge, which reflects the thermodynamical instability of the mixed ensemble. Moreover, specifying the integration contour in (182) would require understanding the singularities of the topological amplitude. These subtleties do not affect the weak form of the

conjecture, since the saddle point approximation to (182) is independent of the details of the contour.

- A variant of the OSV conjecture (182) has been proposed in [58], which involves an integral over both X^I and $\bar{X}^I$, or equivalently a thermodynamical ensemble with fixed electric and magnetic potentials (see Exercise 7 on page 23). It would be interesting to demonstrate the equivalence of this approach with the one based on the holomorphic polarization of the topological amplitude [93].

The OSV conjecture has been successfully tested in the case of non-compact Calabi-Yau manifolds of the form $O(-m)\oplus O(2g-2+m)\to\Sigma_g$, where Σ is a genus g Riemann surface [109, 110]: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m)\to\Sigma$, which is equivalent to 2D Yang Mills (or a q-deformation thereof, when $g\neq 1$). At large N, the partition function of 2D Yang-Mills indeed factorizes into two chiral halves [111], which indeed agree with the topological amplitude computed independently. Exponentially suppressed corrections to the large N limit of 2D Yang-Mills have been studied in [112] and seem to call for a "second quantization" of the right-hand side of (181). For $\mathcal{N}=4$ and $\mathcal{N}=8$ compactifications on $K3\times T^2$ and T^6, the formula (182) has been compared to the prediction for dyons degeneracies based on U-dualities, and agreement has been found in the semi-classical approximation [113]. More recently, several "derivations" of the weak form of the OSV conjecture have been given, using an $M2-\bar{M}2$ or $D6-\bar{D}6$ representation of the black hole, and some modular properties of the partition function [114, 115, 116, 117]. These approaches make it clear that the strong form of the conjecture cannot hold and suggest possible sources of deviations from the "modulus square" form.

In the next section, we shall present a precision test of the OSV conjecture in the context of small black holes in $\mathcal{N}=4$ and $\mathcal{N}=2$ theories, whose microscopic counting can be made exactly.

6 Precision Counting of Small Black Holes

In order to test the OSV conjecture, one should be able to compute subleading corrections to the microscopic degeneracies $\Omega(p,q)$. Due to subtleties in the "black string" CFT description of 4-dimensional black holes, it has not been hitherto possible to reliably compute subleading corrections to (17) for generic BPS black holes.

On the other hand, the heterotic string has a variety of BPS excitations which can be counted exactly using standard wordsheet techniques. Since these states are only charged electrically (in the natural heterotic polarization), their Bekenstein-Hawking entropy evaluated using tree-level supergravity vanishes. This means that higher derivative corrections cannot be neglected, and indeed, upon including R^2 corrections to the effective action, a smooth horizon with finite area is obtained. We refer to these states as "small black holes" to be contrasted with "large black holes" which have non-vanishing entropy already at tree level. This section is based on [5, 6, 118].

6.1 Degeneracies of DH States and the Rademacher Formula

The simplest example to study this phenomenon is the heterotic string compactified on T^6. A class of perturbative BPS states, known as "Dabholkar-Harvey" (DH) states, can be constructed by tensoring the ground state of the right-moving superconformal theory with a level N excitation of the 24 left-moving bosons and adding momentum n and winding w along one circle in T^6 such that the level matching condition $N-1=nw$ is satisfied [119, 120]. The number of distinct DH states with fixed charges (n,w) is $\Omega(n,w)=p_{24}(N)$, where $p_{24}(N)$ is the number of partitions on N into the sum of 24 integers (up to an overall factor of 16 corresponding to the size of short $\mathscr{N}=4$ multiplets, which we will always drop). Accordingly, the generating function of the degeneracies of DH states is

$$\sum_{N=0}^{\infty} p_{24}(N)\, q^{N-1} = \frac{1}{\Delta(q)}, \tag{184}$$

where $\Delta(q)$ is Jacobi's discriminant function

$$\Delta(q) = \eta^{24}(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \tag{185}$$

In order to determine the asymptotic density of states at large $N-1=nw$, it is convenient to extract $d(N)$ from the partition function (184) by an inverse Laplace transform,

$$p_{24}(N) = \frac{1}{2\pi i}\int_{\epsilon-i\pi}^{\epsilon+i\pi} d\beta\; e^{\beta(N-1)}\frac{16}{\Delta(e^{-\beta})}. \tag{186}$$

where the contour C runs from $\epsilon-i\pi$ to $\epsilon+i\pi$, parallel to the imaginary axis. One may now take the high temperature limit $\epsilon\to 0$ and use the modular property of the discriminant function

$$\Delta\left(e^{-\beta}\right) = \left(\frac{\beta}{2\pi}\right)^{-12}\Delta(e^{-4\pi^2/\beta}). \tag{187}$$

As $e^{-4\pi^2/\beta}\to 0$, we can approximate $\Delta(q)\sim q$ and write the integral as

$$p_{24}(N) = \frac{16}{2\pi i}\int_C d\beta\,\left(\frac{\beta}{2\pi}\right)^{12} e^{\beta(N-1)+4\frac{\pi^2}{\beta}} \tag{188}$$

This integral may be evaluated by steepest descent: the saddle point occurs at $\beta = 2\pi/\sqrt{N-1}$, leading to the characteristic Hagedorn growth

$$p_{24}(N) \sim \exp\left(4\pi\sqrt{nw}\right) \tag{189}$$

for the spectrum of DH states.

To calculate the sub-leading terms systematically in an asymptotic expansion at large N, one may recognize that (188) is proportional to the integral representation of a modified Bessel function,

$$\hat{I}_\nu(z) = -i(2\pi)^\nu \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{\nu+1}} e^{(t+z^2/4t)} = 2\pi\,(fracz4\pi)^{-\nu} I_\nu(z) \tag{190}$$

We thus obtain

$$\Omega(n,w) = p_{24}(N) \sim 2^4\, \hat{I}_{13}\left(4\pi\sqrt{nw}\right) . \tag{191}$$

Using the standard asymptotic expansion of $\hat{I}_\nu(z)$ at large z

$$\begin{aligned}\hat{I}_\nu(z) \sim & 2^\nu \left(\frac{z}{2\pi}\right)^{-\nu-\frac{1}{2}} \left[1 - \frac{(\mu-1)}{8z} + \frac{(\mu-1)\left(\mu-3^2\right)}{2!(8z)^2}\right.\\ & \left. - \frac{(\mu-1)\left(\mu-3^2\right)\left(\mu-5^2\right)}{3!(8z)^3} + \dots\right],\end{aligned} \tag{192}$$

where $\mu = 4\nu^2$, we can compute the subleading corrections to the microscopic entropy of DH states to arbitrary high order,

$$\log\Omega(n,w) \sim 4\pi\sqrt{|nw|} - \frac{27}{4}\log|nw| + \frac{15}{2}\log 2 - \frac{675}{32\pi\sqrt{|nw|}} - \frac{675}{2^8\pi^2|nw|} - \dots \tag{193}$$

This is still *not* the complete asymptotic expansion of $\Omega(n,w)$ at large charge. Exponentially suppressed corrections to (191) can be computed by using the Rademacher formula (see [121] for a physicist account)

$$\begin{aligned}F_\nu(n) = \sum_{c=1}^{\infty}\sum_{\mu=1}^{r} c^{w-2}\mathrm{Kl}(n,\nu,m,\mu;c) \sum_{m+\Delta_\mu<0} F_\mu(m)\\ |m+\Delta_\mu|^{1-w}\hat{I}_{1-w}\left[\frac{4\pi}{c}\sqrt{|m+\Delta_\mu|(n+\Delta_\nu)}\right].\end{aligned} \tag{194}$$

In this somewhat formidable expression, $F_\mu(m)$ denote the Fourier coefficients of a vector of modular forms

$$f_\mu(\tau) = q^{\Delta_\mu} \sum_{m\geq 0} F_\mu(m) q^m \qquad \mu = 1,\dots,r \tag{195}$$

which transforms as a finite-dimensional unitary representation of the modular group of weight $w<0$, with

$$f_\mu(\tau+1) = e^{2\pi i\Delta_\mu} f_\mu(\tau) \tag{196}$$

$$f_\mu(-1/\tau) = (-i\tau)^w S_{\mu\nu} f_\nu(\tau) \tag{197}$$

The coefficients $\mathrm{Kl}(n,\nu,m,\mu;c)$ are generalized Kloosterman sums, defined as

$$\mathrm{Kl}(n,\nu;m,\mu;c) \equiv \sum_{0<d<c;d\wedge c=1} e^{2\pi i \frac{d}{c}(n+\Delta_\nu)}\, M(\gamma_{c,d})^{-1}_{\nu\mu}\, e^{2\pi i \frac{a}{c}(m+\Delta_\mu)} \tag{198}$$

where

$$\gamma_{c,d} = \begin{pmatrix} a & (ad-1)/c \\ c & d \end{pmatrix} \tag{199}$$

is an element of $Sl(2,\mathbb{Z})$ and $M(\gamma)$ its matrix representation. For $c=1$ in particular, we have:

$$\mathrm{Kl}(n,\nu,m,\mu;c=1) = S^{-1}_{\nu\mu} \tag{200}$$

Going back to (194), we see that the growth of the Fourier coefficients is determined only by the Fourier coefficients of the "polar" part $F_\mu(m)$ where $m+\Delta_\mu<0$, as well as some modular data. The Ramanujan-Hardy formula

$$F_\mu(n) \sim \exp\left[2\pi\sqrt{\frac{c_{\rm eff}}{6}n}\right] \tag{201}$$

is reproduced by keeping the leading term $c=1$ only, using $\Delta = c_{\rm eff}/24$, $w = -c_{\rm eff}/2$ and the asymptotic behavior (192). The terms with $c>1$ also grow exponentially, but at a slower rate than the term with $c=1$. They therefore contribute exponentially suppressed contributions to $\log F_\nu(n)$.

Applying (194) to the case at hand, we have the convergent series expansion

$$\Omega(n,w) = 2^4 \sum_{c=1}^{\infty} c^{-14}\, \mathrm{Kl}(nw+1,0;c)\, \hat{I}_{13}\left(\frac{4}{c}\pi\sqrt{|nw|}\right) \tag{202}$$

6.2 Macroscopic Entropy and the Topological Amplitude

We now turn to the macroscopic side, and determine the Bekenstein-Hawking-Wald entropy for a BPS black hole with the above charges. Since the attractor formalism is tailored for $\mathcal{N}=2$ supergravity, one should first decompose the spectrum in $\mathcal{N}=2$ multiplets: the $\mathcal{N}=4$ spectrum decomposes into one $\mathcal{N}=2$ gravity multiplet, 2 gravitino multiplets and $n_V=23$ vector multiplets (not counting the graviphoton). Provided the charges under the 4 vectors in the gravitino multiplets vanish, the $\mathcal{N}=2$ attractor mechanism applies.

The topological amplitude F_1 has been computed in [122] and can be obtained as the holomorphic part of the R^2 amplitude at one-loop,

$$f_{R^2} = 24\log\left(T_2|\eta(T)|^4\right) \tag{203}$$

where T, U denote the Kähler and complex structure moduli of the torus T^2. All higher topological amplitudes F_g for $g>1$ vanish for models with $\mathcal{N}=4$ supersymmetry. We therefore obtain the generalized prepotential

$$F(X^I, W^2) = -\frac{1}{2}\sum_{a,b=2}^{23} C_{ab}\frac{X^a X^b X^1}{X^0} - \frac{W^2}{128\pi i}\log\Delta(q) \tag{204}$$

where C_{ab} is the intersection matrix on $H^2(K3)$, $T = X^1/X^0$ and $q = e^{2\pi i T}$. The appearance of the same discriminant function $\Delta(q)$ as in the microscopic heterotic counting (184) is at this stage coincidental.

Identifying $p^1 = w$, $q_0 = n$ and allowing for arbitrary electric charges q_0, $q_{i=2..23}$, the black hole free energy (179) reduces to

$$\mathscr{F}(\phi^I, p^I) = -\frac{\pi}{2}C_{ab}\frac{\phi^a\phi^b p^1}{\phi^0} - \log|\Delta(q)|^2 \tag{205}$$

where

$$q = \exp\left[\frac{2\pi}{\phi^0}\left(p^1 + i\phi^1\right)\right]. \tag{206}$$

The Bekenstein-Hawking-Wald entropy is then obtained by performing a Legendre transform over all electric potentials ϕ^I, $I = 0,\dots 23$. The Legendre transform over $\phi^{a=2..23}$ sets $\phi^a = (\phi^0/p^1)C^{ab}q_b$, where C^{ab} is the inverse of the matrix C_{ab}. We will check *a posteriori* that in the large charge limit, it is consistent to approximate $\Delta(q) \sim q$, whereby all dependence on ϕ^1 disappears. We thus obtain

$$S_{BHW} \sim \left\langle \left[-\frac{\pi}{2}\frac{C^{ab}q_a q_b}{p^1}\phi^0 + 4\pi\frac{p^1}{\phi^0} + \pi\phi^0 q_0\right]\right\rangle_{\phi^0} \tag{207}$$

The extremum of the bracket lies at

$$\phi^0_* = \frac{1}{2}\sqrt{-p^1/\hat{q}_0}\,, \qquad \hat{q}_0 \equiv q_0 + \frac{1}{2p^1}C^{ab}q_a q_b \tag{208}$$

so that at the horizon the Kähler class $\mathrm{Im}T \sim \sqrt{-p^1\hat{q}_0}$ is very large, justifying our assumption. Evaluating (207) at the extremum, we find

$$S_{BH} \sim 4\pi\sqrt{Q^2/2}\,, \quad Q^2 = 2p^1 q_0 + C^{ab}q_a q_b \tag{209}$$

in agreement with the leading exponential behavior in (193), including the precise numerical factor. Note that the argument up to this stage is independent of the OSV conjecture and relies only on the classical attractor mechanism in the presence of higher derivative corrections. The fact that the Bekenstein-Hawking entropy of small black holes comes out proportional to $\sqrt{Q^2/2}$ was argued in [123, 124, 125], based ongeneral scaling arguments. The precise numerical agreement was demonstrated in [118], although with hindsight it could also have been observed by the authors of [38]. This agreement indicates that the tree-level R^2 coupling in the effective action of the heterotic string on T^6 (or, equivalently, large volume limit of the 1-loop R^2 coupling in type IIA/$K3\times T^2$) is sufficient to cloak the singularity of the small black hole behind a smooth horizon. This is in fact confirmed by a study of the corrected geometry [125, 126, 127].

6.3 Testing the OSV Formula

We are now ready to test the proposal (182) and evaluate the inverse Laplace transform of $\exp(\mathscr{F})$ with respect to the electric potentials,

$$\Omega_{OSV}(p) = \int d\phi^0\, d\phi^1\, d^{22}\phi^a \frac{1}{|\Delta(q)|^2} \exp\left[-\frac{\pi}{2} C_{ab} \frac{\phi^a \phi^b p^1}{\phi^0} + \pi\phi^0 q_0 + \pi\phi^a q_a\right] \tag{210}$$

Due to the non-definite signature of C_{ab}, the integral over ϕ^a diverges for real values. This may be avoided by rotating the integration contour to $\epsilon + i\mathbb{R}$ for all ϕs. The integral over ϕ^a is now a Gaussian, leading to

$$\Omega_{OSV}(Q) = \int d\phi^0\, d\phi^1 \left(\frac{\phi_0}{p^1}\right)^{11} \frac{1}{|\Delta(q)|^2} \exp\left(-\frac{1}{2}\frac{C^{ab} q_a q_b}{p^1}\phi^0 + q_0\phi^0\right) \tag{211}$$

where we dropped numerical factors and used the fact that $\det C = 1$. The asymptotics of Ω is independent of the details of the contour, as long as it selects the correct classical saddle point (208) at large charge. Approximating again $\Delta(q) \sim q$, we find the quantum version of (207),

$$\Omega_{OSV}(Q) \sim \int d\phi^0\, d\phi^1 \left(\frac{\phi_0}{p^1}\right)^{11} \exp\left(-\frac{1}{2}\frac{C^{ab} q_a q_b}{p^1}\phi^0 - 4\pi\frac{p^1}{\phi^0} + q_0\phi^0\right) \tag{212}$$

The integral over ϕ^1 superficially leads to an infinite result. However, since the free energy is invariant under $\phi^1 \to \phi^1 + \phi^0$, it is natural to restrict the integration to a single period $[0,\ \phi^0]$, leading to an extra factor of ϕ^0 in (212). The integral over ϕ^0 is now of Bessel type, leading to

$$\Omega_{OSV}(Q) = (p^1)^2\, \hat{I}_{13}\left(4\pi\sqrt{Q^2/2}\right) \tag{213}$$

in impressive agreement with the microscopic result (191) at all orders in $1/Q$.

Some remarks on this computation are in order:

- Note that the extra factor $(p^1)^2$ in (213) is inconsistent with $SO(6,22,\mathbb{Z})$ duality, which requires the exact degeneracies to be a function of Q^2 only. Moreover, the agreement depends crucially on discarding the non-holomorphic correction proportional to $\log T_2$ in F_1. Both of these issues call for a better understanding of the relation between the physical amplitude and the topological wave function in the real polarization. It should be mentioned that an alternative approach has been developped by Sen, keeping the non-holomorphic corrections but using a different statistical ensemble [128, 129].
- The "all order" result (213) depends only on the number of $\mathcal{N}=2$ vector multiplets, as well as on the leading large volume behavior of $F_1 \sim q/(128\pi i)$. By heterotic/type II duality, this term is mapped to a tree-level R^2 interaction on the heterotic side, which is in fact universal. We thus conclude that in all $\mathcal{N}=2$

models which admit a dual heterotic description, provided higher genus $F_{g>1}$ and genus 0,1 Gromov-Witten instantons can be neglected, the degeneracies of small black holes predicted by (182) are given by

$$\Omega_{OSV}(Q) \propto \hat{I}_{\frac{n_V+3}{2}}\left(4\pi\sqrt{Q^2/2}\right) \,, \tag{214}$$

where n_V is the number of Abelian gauge fields, including the graviphoton. We return to the validity of the assumption in the next subsection.

Exercise 17. *Applying a similar argument to large black holes with $p^0 = 0$, assuming that only the large-volume limit of F_1 contributes, shows that the OSV conjecture (182), in the saddle point approximation, predicts [5, 6]*

$$\Omega(p^A, q_A) \sim \pm\frac{1}{2}|\det C_{ab}(p)|^{-1/2}\left(\hat{C}(p)/6\right)^{\frac{n_V+2}{2}} \times \hat{I}_{\frac{n_V+2}{2}}\left(2\pi\sqrt{-\hat{C}(p)\hat{q}_0/6}\right) \tag{215}$$

where

$$C_{AB}(p) = C_{ABC}p^C, \quad C(p) = C_{ABC}p^A p^B p^C, \quad \hat{C}(p) = C(p) + c_{2A}p^A \,, \tag{216}$$

and compare to the microscopic counting (19).

- In order to see whether the strong version of the OSV conjecture has a chance to hold, it is instructive to change variable to $\beta = \pi/t$ in (186) and rewrite the exact microscopic result as

$$\Omega_{\text{exact}}(Q) = \int dt\, t^{-14}\, \frac{\exp\left(\frac{\pi n w}{t}\right)}{\Delta\left(e^{-4\pi t}\right)} \tag{217}$$

  On the other hand, it is convenient to change variables in the OSV integral (211) to $\tau_1 = \phi^1/\phi^0,\ \tau_2 = -p^1/\phi^0$, with Jacobian $d\phi^0 d\phi^1 = 8(p^1)^2 d\tau_1 d\tau_2/\tau_2^3$, leading to

$$\Omega_{OSV}(Q) \sim \int d\tau_1\, d\tau_2\, \tau_2^{-14} \frac{\exp\left(\frac{\pi n w}{\tau_2}\right)}{|\Delta\left(e^{-2\pi\tau_2 + 2\pi i \tau_1}\right)|^2} \tag{218}$$

  Despite obvious similarities, it appears unlikely that the two results are equal non-perturbatively.
- Just as the perturbative result (191), the result (213) misses subleading terms in the Rademacher expansion (202). It does not seem possible to interpret any of the terms with $c > 1$ as the contribution of a subleading saddle point in either (188) or (211).

Despite these difficulties, it is remarkable that the black hole partition function in the OSV ensemble, obtained from purely macroscopic considerations, reproduces the entire asymptotic series exactly to all orders in inverse charge. Recent developments show that this agreement is largely a consequence of supersymmetry and anomaly cancelation for black holes which have an AdS_3 region [107, 130, 131] (see also the lectures by P. Kraus [132] in this volume).

6.4 $\mathscr{N}$ = 2 Orbifolds

We conclude this section with a few words on small black holes in $\mathscr{N} = 2$ orbifolds, referring to [5, 6] for detailed computations. We find that the agreement found in $\mathscr{N} = 4$ models broadly continues to hold, with the following caveats:

- In contrast to $\mathscr{N} = 4$ cases, the neglect of Gromov-Witten instantons is harder to justify rigorously: when $\chi(X) \neq 0$, the series of point-like instantons contribution becomes strongly coupled in the regime of validity of the Rademacher formula, $\hat{q}_0 \gg \hat{C}(p)$. The strong coupling behavior is controlled, up to a logarithmic term, by the Mac-Mahon function (135), which is exponentially suppressed in this regime. The logarithmic term in (139) may be reabsorbed into a redefinition of the topological string amplitude $\Psi_{\text{top}} \to \lambda^{\chi/24} \Psi_{\text{top}}$. As for non-degenerate instantons, they are exponentially suppressed provided all magnetic charges are non zero. This is unfortunately not the case for the small black holes dual to the heterotic DH states, whose Kähler classes are attracted to the boundary of the Kähler cone at the horizon.
- For BPS states in twisted sectors of $\mathscr{N} = 2$ orbifolds, we find that the instanton-deprived OSV proposal appears to successfully reproduce the *absolute degeneracies*, equal to the indexed degeneracies, to all orders. For untwisted DH states of the OSV prediction appears to agree with the *absolute degeneracies* of untwisted DH states to leading order (which have the same exponential growth as twisted DH states) but not at subleading order (as the subleading corrections in the untwisted sector are moduli-dependent, and uniformly smaller than in the twisted sectors). The indexed degeneracies are exponentially smaller than absolute degeneracies due to cancelations of pairs of DH states, so plainly disagree with the OSV prediction.

7 Quantum Attractors and Automorphic Partition Functions

In this final chapter, we elaborate on an intriguing proposal by Ooguri, Verlinde, and Vafa [7] to interpret the OSV conjecture as a holographic duality between the usual Hilbert space of black hole micro-states quantized with respect to global time, and the Hilbert space of stationary, spherically symmetric geometries quantized with respect to the radial direction. Although we shall find some difficulties in implementing this proposal literally, this line of thought will prove fruitful in suggesting non-perturbative extensions of the OSV conjecture. In particular, we shall find tantalizing hints of a one-parameter generalization of the topological string amplitude in $\mathscr{N} = 2$ theories, and obtain a natural framework for constructing automorphic black hole partition functions (in cases with suitably large U-duality groups) which go beyond the Siegel modular forms discussed in Sect. 2.5. This chapter is based on [8, 9, 10, 11, 12, 13].

7.1 OSV Conjecture and Quantum Attractors

In order to motivate this approach, recall that, after analytically continuing $\phi^I = i\chi^I$ to the imaginary axis, the right-hand side of the OSV conjecture (182)

$$\Omega\left(p^I, q_I\right) \sim \int d\chi^I \Psi^*_{\text{top}}(p^I + \chi^I)\Psi_{\text{top}}\left(p^I - \chi^I\right) e^{i\pi\chi^I q_I} \equiv W_{\Psi_{\text{top}}}(p^I, q_I) \qquad (219)$$

could be interpreted as the Wigner distribution associated to the wave function Ψ_{top}. In usual quantum mechanics, the Wigner distribution $W_\psi(p,\ q)$ is a function on phase space associated to a wave function $\psi(q)$, such that quantum averages of Weyl-ordered operators on ψ are equal to classical averages of their symbols with respect to W_ψ,

$$\langle\psi|\mathscr{O}(\hat{p},\hat{q})|\psi\rangle = \int dp\, dq\, W_\psi(p,q)\ \mathscr{O}(p,q) \qquad (220)$$

Moreover, when ψ satisfies the Schrödinger equation, W satisfies the classical Liouville equation to leading order in $\hbar$; the Wigner distribution is thus a useful tool to study the semi-classical limit. The above observation thus begs the question: What is the physical quantum system of which Ψ_{top} is the wave function[19], and how does it encode the black hole degeneracies?

Exercise 18. *Show that*

$$W_{\tilde{\psi}}(p^I, q_I) = W_\psi\left(\frac{q_I}{2}, 2p^I\right) \qquad (221)$$

where $\tilde{\psi}(\phi) = \int d\chi\ e^{i\pi\chi\phi}\ \psi(\chi)$ is the Fourier transform of ψ.

In order to try and answer this question, it is useful to reabsorb the dependence on the charges (p^I, q_I) into the state itself, by defining

$$\Psi^\pm_{p,q}(\chi) \equiv e^{\pm i\pi q\chi}\Psi^\pm_{\text{top}}(\chi \mp p) \equiv V^\pm_{p,q}\cdot\Psi_{\text{top}}(\chi) \qquad (222)$$

Equation (220) is then rewritten more suggestively as an overlap of two wave functions,

$$\Omega(p,q) \sim \int d\chi\ [\Psi^-]^*_{p,q}(\chi)\ \Psi^+_{p,q}(\chi) \qquad (223)$$

On the other hand, recall that the near horizon geometry $AdS_2 \times S^2$, written in global coordinates as

[19] Or, to paraphrase Ford Prefect, what is the Question to the Answer Ψ_{top}?

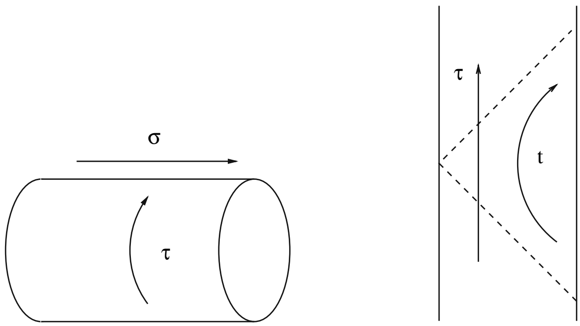

Fig. 3 *Left*: the cylinder amplitude in string theory can be viewed either as a trace over the open string Hilbert space (quantizing along τ channel) or as an inner product between two wave functions in the closed string Hilbert space (quantizing along σ). *Right*: The global geometry of Lorentzian AdS_2 has the topology of a strip; its Euclidean continuation at finite temperature becomes a cylinder. τ and t are the global and Poincaré time, respectively

$$ds^2 = |Z_*|^2 \left(\frac{-d\tau^2 + d\sigma^2}{\cos^2 \sigma} + d^2\Omega \right) \tag{224}$$

has two distinct conformal boundaries at $\sigma = 0, \ \pi$, respectively; its Euclidean sections at finite temperature have the topology of a cylinder (see Fig. 3).

Exercise 19. *Check that the metric (224) is equivalent to (13) upon changing coordinates* $\tau = \arctan(z+t) - \arctan(z-t), \quad \sigma = \arctan(z+t) + \arctan(z-t)$. *Map out the portion of the global geometry covered by the Poincaré coordinates* $z, \ t$.

With this in mind, it is tempting to view (223) as an analogue of open/closed duality for conformal field theory on the cylinder,

$$\mathrm{Tr}\, e^{-\pi t H_{\mathrm{open}}} = \langle B' | e^{-\frac{\pi}{t} H_{\mathrm{closed}}} | B \rangle \tag{225}$$

where $|B\rangle$ and $|B'\rangle$ are closed string boundary states. The right-hand side of (223), analogue of the closed string channel, is identified with the partition function of quantum gravity on $AdS_2 \times S^2$ in radial quantization along the space-like coordinate σ, with boundary conditions at $\sigma = 0, \ \pi$ specified by the "boundary states" $\Psi^{\pm}_{p,q}$, while the left-hand side, analogue of the open string channel, is recognized as a trace of the identity operator in a sector of the Hilbert space for quantization along the global time coordinate τ, with fixed charges p^I, q_I (the absence of an analogue of the Hamiltonians H_{open} and H_{closed} can be traced to diffeomorphism invariance, which requires physical states to be solutions of the Wheeler-De Witt equation $H|\psi\rangle = 0$). It should be stressed that the Hilbert spaces for time-like and radial quantization are distinct, just like the open string and closed string Hilbert spaces are different.

For this interpretation to make sense, it should of course be possible to view Ψ_{top} as a state in the Hilbert space for radial quantization. This is, at least superficially, consistent with the wave function interpretation of Ψ_{top} discussed in Sect. 4.4 and would in fact provide a nice physical interpretation of this otherwise mysterious quantum mechanical behavior. Moreover, the functional dimension, n_V+1, of the Hilbert space hosting $\Psi_{\rm top}$, is roughly in accordance with the number of complex scalars z^i varying radially in the black hole geometry. This leads one to expect that $\Psi_{\rm top}$ may provide a radial wave function for the vector-multiplet scalars in a truncated Hilbert space where only static, spherically symmetric BPS configurations are kept. Such a "mini-superspace" truncation is usually hard to justify but may hopefully be suitable for the purpose of computing indexed degeneracies of BPS black holes, in the same way as the Ramond-Ramond ground states in the closed string channel control the growth of the index in the open string Ramond sector.

This brings us to the problems of (i) quantizing the attractor flow (53), (54), (ii) showing that the resulting Hilbert space is the correct habitat for $\Psi_{\rm top}$, and (iii) finding a physical principle that selects $\Psi_{\rm top}$ among the continuum of states in that BPS Hilbert space. Answering these questions will be the subject of the rest of this chapter. Before doing so, several general remarks are in order:

- The idea of radial quantization of static black holes has a long history in the canonical gravity literature, e.g. [133, 134, 135, 136, 137, 138]. The main new ingredients here are supersymmetry, which may provide a better justification for the mini-superspace approximation, and holography, which offers the possibility to reconstruct the spectrum of the global time Hamiltonian from the overlap of two radial wave functions. The quantization of BPS configurations has been considered recently in various set-ups and found to agree with gauge theory computations [139, 140, 141, 142, 143, 144].
- The "channel duality" argument is in line with the usual AdS/CFT philosophy that the black hole micro-states should be described by "gauge theoretical" degrees of freedom living on the boundary of AdS_2. Contrary to higher dimensional AdS spaces, the conformal quantum mechanics describing AdS_2 is still largely mysterious, and the above approach is a possible indirect route towards determining its spectrum.
- One usually assumes that black hole micro-states can be described only in terms of the near horizon geometry. The above proposal to quantize the whole attractor flow seems to be at odds with this idea. A possible way out is that the topological wave function be a fixed point of the quantum attractor flow. In the sequel, we will study the full quantum attractor flow, from asymptotic infinity to the horizon, as a function of the Poincaré radial coordinate r (rather than the "global radial coordinate σ", which is well defined only near the horizon).
- The analogy between global AdS_2 and open strings explained below (225) can be pushed quite a bit further: due to pair production of charged particles, a black hole may fragment in different throats, analogous to the joining and splitting interactions of open strings [145] (see [146] for a perturbative approach to this problem). The study of exponentially suppressed corrections to the partition function in certain non-compact Calabi-Yau threefolds suggests that the attractor flow should be "second quantized" to allow for this possibility [112]. Note that the process whereby two ends of an open string join to form a closed string does not seem to have a black hole analogue.

- Finally, let us mention that further interest for the quantization of attractor flows stems from the relation between black hole attractor equations and the equations that determine supersymmetric vacua in flux compactifications (see e.g. [26] for a recent discussion). Upon double analytic continuation, one may hope to relate the black hole wave function to the wave function of the Universe and address vacuum selection in the Landscape [7]. There are however many difficulties with this idea that we shall not discuss here. At any rate, it will be clear that our discussion of radial quantization bears many similarities with "mini-superspace" approaches to quantum cosmology.

7.2 Attractor Flows and Geodesic Motion

The most convenient route to quantize the attractor flow, or more generally perform the radial quantization of stationary, spherically symmetric black holes, is to use the equivalence between the equations governing the radial evolution of the fields in four dimensions, and the geodesic motion of a fiducial particle on an appropriate pseudo-Riemannian manifold [147]. This equivalence holds irrespective of supersymmetry, so we consider the general two-derivative action for four-dimensional gravity coupled to scalar fields z^i and gauge fields A_4^I,

$$S_4 = \frac{1}{2}\int \left[\sqrt{-\gamma}\, R[\gamma]\, d^4x + g_{ij}\, dz^i \wedge \star dz^j - F^I \wedge \left(t_{IJ} \star F^J + \theta_{IJ} \wedge F^J\right)\right]. \quad (226)$$

Here, γ denotes the four-dimensional metric, g_{ij} the metric on the moduli space $\mathscr{M}_4$ where the (real) scalars z^i take their values, $F^I = dA_4^I$ and the (positive definite) gauge couplings t_{IJ} and angles θ_{IJ} are in general functions of z^i. In (226), we have dropped the contribution of the fermionic fields, but we shall reinstate them in Sect. 7.3 below when we return to a supersymmetric setting. Moreover, since the pseudo-Riemannian manifold already arises under the sole assumption of stationarity, we begin by relaxing the assumption of spherical symmetry.

7.2.1 Stationary Solutions and KK^* Reduction

A general ansatz for stationary metrics and gauge fields is

$$\gamma_{\mu\nu} dx^\mu dx^\nu = -e^{2U}(dt+\omega)^2 + e^{-2U}\gamma_{\rm ij} dx^{\rm i} dx^{\rm j}\,, \quad A_4^I = \zeta^I dt + A_3^I\,. \quad (227)$$

where the three-dimensional metric $\gamma_{\rm ij}$, one-forms A_3^I, ω and scalar U, ζ^I, z^i, are general functions of the coordinates $x^{\rm i}$ on the three-dimensional spatial slice. Since all these fields are independent of time, one may reduce the four-dimensional action (226) along the time direction and obtain a field theory in three Euclidean dimensions. This process is analogous to the usual Kaluza-Klein reduction, except for the time-like signature of the Killing vector ∂_t, which leads to unusual sign changes in the three-dimensional action.

Just as in usual Kaluza-Klein reduction, the one-forms A_3^I and ω can be dualized into axionic scalars $\tilde{\zeta}_I$, σ, using Hodge duality between one-forms and pseudo-scalars in three dimensions. Thus, the four-dimensional theory reduces to a gravity-coupled non-linear sigma model

$$S_3 = \frac{1}{2}\int \left(\sqrt{g_3}\, R[g_3]\, d^3x + g_{ab}\, d\phi^a \wedge \star d\phi^b\right) \tag{228}$$

whose target manifold $\mathscr{M}_3^*$ includes the four-dimensional scalar fields z^i together with U, ζ^I, $z\tilde{e}ta_I$, σ. The metric g_{ab} on $\mathscr{M}_3^*$ has indefinite signature and can be obtained by analytic continuation $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$ [147, 148] from the (Riemannian) three-dimensional moduli space $\mathscr{M}_3$ arising in standard, spacelike Kaluza-Klein reduction (see e.g. [149])

$$\begin{aligned} ds^2_{\mathscr{M}_3^*} =& 2\,dU^2 + g_{ij}\,dz^i dz^j + \frac{1}{2}e^{-4U}\left(d\sigma + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2 \\ & - e^{-2U}\left[t_{IJ}\, d\zeta^I d\zeta^J + t^{IJ}\left(d\tilde{\zeta}_I + \theta_{IK} d\zeta^K\right)\left(d\tilde{\zeta}_J + \theta_{JL} d\zeta^L\right)\right] \end{aligned} \tag{229}$$

where $t^{IJ} \equiv [t^{-1}]^{IJ}$. Importantly, $\mathscr{M}_3^*$ always possesses (at least) $2n+2$ isometries corresponding to the gauge symmetries of $A^I, \tilde{A}_I, \omega$, as well as rescalings of time t. The Killing vector fields generating these isometries read

$$p^I = \partial_{\tilde{\zeta}_I} - \zeta^I \partial_\sigma\ , \quad q_I = -\partial_{\zeta^I} - \tilde{\zeta}_I \partial_\sigma\ , \quad k = \partial_\sigma\ , \tag{230a}$$

$$M = -\left(\partial_U + \zeta^I \partial_{\zeta^I} + \tilde{\zeta}^I \partial_{\tilde{\zeta}^I} + 2\sigma\partial_\sigma\right) \tag{230b}$$

and satisfy the Lie-bracket algebra

$$[p^I, q_J] = -2\delta^I_J\, k \tag{231a}$$

$$[M, p^I] = p^I\ , \quad [M, q_I] = q_I\ , \quad [M, k] = 2k \tag{231b}$$

In general, stationary solutions in four dimensions are therefore given by harmonic maps from the three-dimensional slice, with metric $\gamma_{\rm ij}$, to the three-dimensional moduli space $\mathscr{M}_3^*$, such that Einstein's equation in three-dimension is fulfilled,

$$R_{\rm ij}[\gamma] = g_{ab}\left(\partial_{\rm i}\phi^a \partial_{\rm j}\phi^b - \frac{1}{2}\partial_{\rm k}\phi^a\, \partial_{\rm l}\phi^b\, \gamma^{\rm kl}\, \gamma_{\rm ij}\right) \tag{232}$$

Moreover, the Killing vectors p^I, q_I, k, M give rise to conserved currents, whose conserved charges will be identified with the overall electric and magnetic charges, NUT charge and ADM mass of the configuration.

7.2.2 Spherical Symmetry and Geodesic Motion

Now, let us restrict to spherically symmetric, stationary solutions: The spatial slices can be parameterized as

$$\gamma_{\mathrm{ij}} dx^{\mathrm{i}} dx^{\mathrm{j}} = N^2(\rho)\, d\rho^2 + r^2(\rho)\, \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{233}$$

while all scalars on $\mathscr{M}_3^*$ become functions of ρ only. After dropping a total derivative term, the three-dimensional sigma-model action reduces to classical mechanics,

$$S_1 = \int d\rho \left[\frac{N}{2} + \frac{1}{2N}\left(r'^2 - r^2\, g_{ab}\, \phi'^a\, \phi'^b\right)\right] \tag{234}$$

where the prime denotes a derivative with respect to ρ. This Lagrangian describes the free motion of a fiducial particle on a cone[20] $\mathscr{C} = \mathbb{R}^+ \times \mathscr{M}_3^*$ over the three-dimensional moduli space $\mathscr{M}_3$. The lapse N is an auxiliary field; its equation of motion enforces the mass shell condition

$$r'^2 - r^2\, g_{ab}\, \phi'^a\, \phi'^b = N^2 \tag{235}$$

or equivalently, the Wheeler-De Witt (or Hamiltonian) constraint

$$H_{\mathrm{WDW}} = (p_r)^2 - \frac{1}{r^2} g^{ab}\, p_a p_b - 1 \equiv 0 \tag{236}$$

where p_r, p_a are the canonical momenta conjugate to r, ϕ^a.

Solutions are thus massive geodesics on the cone, with fixed mass equal to 1. In particular, the phase space describing the set of stationary, spherically symmetric solutions of (226) is the cotangent bundle $T^*\mathscr{C}$ of the cone $\mathscr{C}$.

As is most easily seen in the gauge $N = r^2$, the motion separates into geodesic motion on the base of the cone $\mathscr{M}_3^*$, with affine parameter τ such that $d\tau = d\rho/r^2(\rho)$, and motion along the radial direction r,

$$(p_r)^2 - \frac{C^2}{r^2} - 1 \equiv 0\,, \quad g^{ab}\, p_a p_b \equiv C^2 \tag{237}$$

where $p_r = r' = \dot{r}/r^2$ and $p_i = r^2 \phi'^i = \dot{\phi}^i$; here the dot denotes a derivative with respect to τ. It is interesting to note that the radial motion is governed by the same Hamiltonian as in [152] and therefore exhibits one-dimensional conformal invariance. This is a consequence of the existence of the homothetic Killing vector $r\partial_r$ on the cone $\mathscr{C}$.

7.2.3 Extremality and Light-Like Geodesics

The motion along r is easily integrated to

$$r = \frac{C}{\sinh(C\tau)}\,, \quad \rho = \frac{C}{\tanh C\tau} \tag{238}$$

[20] A similar mechanical arises in mini-superspace cosmology [150, 151].

Assuming that the sphere S^2 reaches a finite area A at the horizon $\tau = \infty$ so that $e^{-2U} r^2 \to A/(4\pi)$, one may rewrite the metric (227) as [27]

$$ds^2 \sim \frac{C^2}{\sinh^2(C\tau)} \left(-\frac{4\pi}{A}(dt+\omega)^2 + \frac{A}{4\pi} d\tau^2 \right) + \frac{A}{4\pi} d^2\Omega \tag{239}$$

The horizon at $\tau = \infty$ is degenerate for $C^2 = 0$, and non-degenerate for $C^2 > 0$, corresponding to extremal and non-extremal black holes, respectively. We conclude that extremal black holes correspond to *light-like* geodesics on $\mathcal{M}_3^*$ (it is indeed fortunate that $\mathcal{M}_3^*$ is a pseudo-Riemannian manifold so that light-like geodesics do exist).

Exercise 20. *Show that the extremality parameter C is related to the Bekenstein–Haw-king entropy and Hawking temperature by $C = 2S_{BH}T_H$.*

Setting $C = 0$ in (237), we moreover see that $r = \rho = 1/\tau$, and therefore that the spatial slices in the ansatz (227) are flat. We could therefore have set $N = 1$, $r = 1/\tau$ from the start, and obtained the action for geodesic motion on $\mathcal{M}_3$ in affine parameterization,

$$S_1' = \int d\tau \, \frac{1}{2} \, g_{ab} \, \dot{\phi}^a \, \dot{\phi}^b \tag{240}$$

While one may dispose of the radial variable r altogether, it is however advantageous to retain it for the purpose of defining observables such as the horizon area, $A_H = 4\pi e^{-2U} r^2|_{U \to -\infty}$ and the ADM mass $M = r(e^{2U} - 1)|_{U \to 0}$.

7.2.4 Conserved Charges and Black Hole Potential

As anticipated by the notation in (7.13a), the isometries of $\mathcal{M}_3$ imply conserved Noether charges,

$$\begin{aligned} q_I \, d\tau &= -2e^{-2U} \left[t_{IJ} d\zeta^J + \theta_{IJ} t^{JL} \left(d\tilde{\zeta}_L + \theta_{LM} d\zeta^M \right) \right] + 2k\tilde{\zeta}_I \\ p^I \, d\tau &= -2e^{-2U} t^{IL} \left(d\tilde{\zeta}_L + \theta_{LM} d\zeta^M \right) - 2k\zeta^I \\ k \, d\tau &= e^{-4U} \left(d\sigma + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I \right) \end{aligned} \tag{241}$$

(as well as M, whose precise form we will not need) identified as the electric, magnetic and NUT charges p^I, q_I, k. Their algebra under Poisson bracket is the same as algebra of the Killing vectors under Lie bracket,

$$\{p^I, q_J\}_{\rm PB} = -2\delta^I_J k \,, \quad \{M, p^I\}_{\rm PB} = p^I \,, \quad \{M, q_I\}_{\rm PB} = q_I \,, \quad \{M, k\}_{\rm PB} = 2k \tag{242}$$

In particular, the electric and magnetic charges satisfy an Heisenberg algebra, the center of which is the NUT charge k. The latter is related to the off-diagonal term in the metric (227) via $\omega = k \cos\theta \, d\phi$. When $k \neq 0$, the metric

$$ds_4^2 = -e^{2U}(dt + k\cos\theta\, d\phi)^2 + e^{-2U}\left(d\rho^2 + r^2(\rho)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \qquad (243)$$

has closed timelike curves along the compact ϕ coordinates near $\theta = 0$, all the way from infinity to the horizon. Bona fide 4D black holes have $k = 0$, which corresponds to a "classical" limit of the Heisenberg algebra (242).

Using the conserved charges (241), one may express the Hamiltonian for affinely parameterized geodesic motion on $\mathscr{M}_3^*$ as

$$H \equiv p^a g_{ab} p^b = \frac{1}{2}\left[p_U^2 + \frac{1}{4} p_{z^i} g^{ij} p_{z^j} - e^{2U} V_{BH} + k^2 e^{4U}\right] \qquad (244)$$

where $p_U,\ p_{z^i}$ are the momenta canonically conjugate to U, z^i,

$$V_{BH}(p,q,z) = -\frac{1}{2}\left(\hat{q}_I - \theta_{IJ}\hat{p}^J\right) t^{IK}\left(\hat{q}_K - \theta_{KL}\hat{p}^L\right) - \frac{1}{2}\hat{p}^I t_{IJ}\hat{p}^J \qquad (245)$$

and

$$\hat{p}^I = p^I + 2k\zeta^I\,, \quad \hat{q}_I = q_I - 2k\zeta^I\,, \qquad (246)$$

For $k = 0$, the motion along $\zeta^I, \tilde{\zeta}_I, \sigma$ separates from that along U, z^i, effectively producing a potential for these variables. Following [55], we refer to V_{BH} as the "black hole potential", but it should be kept in mind that it contributes negatively to the actual potential $V = -e^{2U} V_{BH} + k^2 e^{4U}$ governing the Hamiltonian motion. In Fig. 4, we plot the potential V for $\mathscr{N} = 2$ supergravity with one minimally coupled vector multiplet.

7.2.5 The Universal Sector

As an illustration, and a useful warm-up for the symmetric case discussed in Sect. 7.5 below, it is instructive to work out the dynamics in the "universal sector",

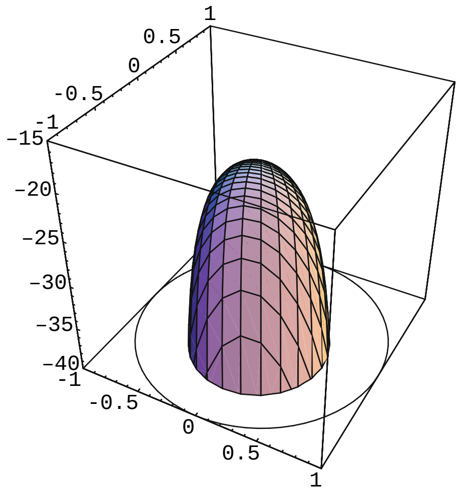

Fig. 4 Potential governing the radial evolution of the complex scalar in the same model as in Fig. 2 and same charges, at $U = 0$. The potential has a global maximum at $z_* = X^1/X^0 = (1-3i)/10$

which encodes the scale U, the graviphoton electric and magnetic charges, and the NUT charge k. The resulting pseudo-quaternionic-Kähler manifold is the symmetric space $\mathscr{M}_3^* = SU(2,1)/Sl(2)\times U(1)$, an analytic continuation of the quaternionic-Kähler space $\mathscr{M}_3 = SU(2,1)/SU(2)\times U(1)$, which describes the tree-level couplings of the universal hypermultiplet in 4 dimensions. It is obtained via c-map from a trivial moduli space $\mathscr{M}_4$ corresponding to the prepotential $F=-i(X^0)^2/2$. The Hamiltonian (244) becomes

$$H = \frac{1}{8}(p_U)^2 - \frac{1}{4}e^{2U}\left[(p_{\tilde{\zeta}} - k\zeta)^2 + (p_\zeta + k\tilde{\zeta})^2\right] + \frac{1}{2}e^{4U}k^2 \tag{247}$$

The motion separates between the $(\tilde{\zeta},\zeta)$ plane and the U direction, while the NUT potential σ can be eliminated in favor of its conjugate momentum $k = e^{-4U}(\dot{\sigma} + \zeta\dot{\tilde{\zeta}} - \tilde{\zeta}\dot{\zeta})$. The motion in the $(\tilde{\zeta},\zeta)$ plane is that of a charged particle in a constant magnetic field. The electric, magnetic charges, and the angular momentum J in the plane (not to be confused with that of the black hole, which vanishes by spherical symmetry)

$$p = p_{\tilde{\zeta}} + \zeta k, \quad q = p_\zeta - \tilde{\zeta}k, \quad J = \zeta p_{\tilde{\zeta}} - \tilde{\zeta} p_\zeta \tag{248}$$

satisfy the usual algebra of the Landau problem,

$$\{p,q\}_{\rm PB} = 2k,\ \{[J,p\}_{\rm PB} = q, \quad \{J,q\}_{\rm PB} = -p \tag{249}$$

where p and q are the "magnetic translations". The motion in the U direction is governed effectively by

$$H = \frac{1}{8}(p_U)^2 + \frac{1}{2}e^{4U}k^2 - \frac{1}{4}e^{2U}\left[p^2+q^2-4kJ\right] = C^2 \tag{250}$$

The potential is depicted on Fig. 5 (left). At spatial infinity $(\tau=0)$, one may impose the initial conditions $U=\zeta=\tilde{\zeta}=a=0$. The momentum p_U at infinity is proportional to the ADM mass, and J vanishes, so the mass shell condition (247) becomes

$$M^2 + 2k^2 - \left(p^2+q^2\right) = C^2 \tag{251}$$

Extremal black holes correspond to $C^2=0$; in this low dimensional example are automatically BPS, as we shall see in the next section. Equation (251) is then the BPS mass condition, generalized to non-zero NUT charge. Note that for a given value of p,q, there is a maximal value of k such that M^2 remains positive.

At the horizon $U\to-\infty,\ \tau\to\infty$, the last term in (247) is irrelevant, and one may integrate the equation of motion of U, and verify that the metric (227) becomes $AdS_2\times S^2$ with area

$$A = 2\pi(p^2+q^2) = 2\pi\sqrt{(p^2+q^2)^2} \tag{252}$$

in agreement with the Bekenstein-Hawking entropy of Reissner-Nordström black holes (15).

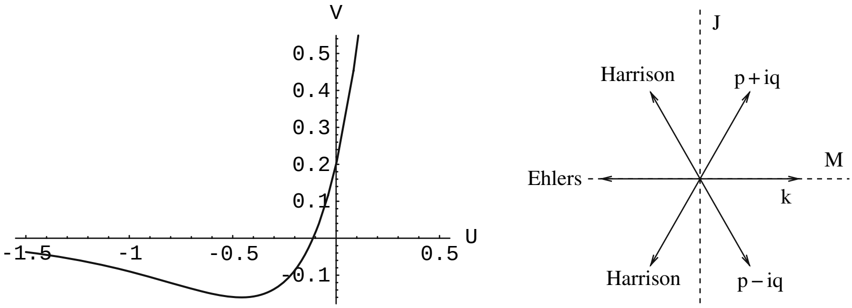

Fig. 5 *Left*: Potential governing the motion along the U variable in the universal sector. The horizon is reached at $U \to -\infty$. *Right*: Root diagram of the $SU(2,1)$ symmetries in the universal sector

Since the universal sector is a symmetric space, there must exist 3 additional conserved charges, so that the total set of conserved charges can be arranged in an element Q in the Lie algebra $\mathfrak{g}_3 = su(2,1)$ (or rather, in its dual $\mathfrak{g}_3^*$),

$$Q = \begin{pmatrix} M + iJ/3 & E_p - iE_q, & iE_k \\ E_{p'} + iE_{q'} & -2iJ/3 & -(E_p + iE_q) \\ -iE_{k'} & -(E_{p'} - iE_{q'}) & -M + iJ/3 \end{pmatrix} \tag{253}$$

where $M, E_p \equiv p^0, E_q \equiv q_0, E_k \equiv k$ have been given in (7.12a) and $J = \zeta\partial_{\tilde{\zeta}} - \tilde{\zeta}\partial_{\zeta}$. The remaining Killing vectors can be easily found [13],

$$\begin{aligned}
E_{p'} &= -\tilde{\zeta}\partial_U - (\sigma + 2\zeta\tilde{\zeta})\partial_\zeta + \left[e^{2U} + \frac{1}{2}(3\zeta^2 - \tilde{\zeta}^2)\right]\partial_{\tilde{\zeta}} + \left[\zeta\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)\right) - \sigma\tilde{\zeta}\right]\partial_\sigma \\
E_{q'} &= \zeta\partial_U - \left[e^{2U} + \frac{1}{2}(3\tilde{\zeta}^2 - \zeta^2)\right]\partial_\zeta - (\sigma - 2\zeta\tilde{\zeta})\partial_{\tilde{\zeta}} + \left[\tilde{\zeta}\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)\right) + \sigma\zeta\right]\partial_\sigma \\
E_{k'} &= -\sigma\partial_U + \left[\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)\right)^2 - \sigma^2\right]\partial_\sigma \\
&\quad - \left[\tilde{\zeta}\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)\right) + \sigma\zeta\right]\partial_\zeta + \left[\zeta\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)\right) - \sigma\tilde{\zeta}\right]\partial_{\tilde{\zeta}}
\end{aligned}$$

The physical origin of these extra symmetries are the Ehlers and Harrison transformations, well known to general relativists [153]. It is easy to check that these Killing vectors satisfy the Lie algebra of $SU(2,1)$, whose root diagram is depicted on Fig. 5. The Casimir invariants of Q can be easily computed:

$$\mathrm{Tr}(Q^2) = H\ , \quad \det(Q) = 0 \tag{254}$$

The last condition ensures that the conserved quantities do not overdetermine the motion. The co-adjoint action $Q \to hQh^{-1}$ of G_3 on $\mathfrak{g}_3^*$ relates different trajectories with the same value of H. The phase space, at fixed value of H, is therefore a generic co-adjoint orbit of G_3, of dimension 6 (the symplectic quotient of the full 8-dimensional phase space by the Hamiltonian H). By the Kirillov-Kostant construction, it carries a canonical symplectic form such that the Noether charges represent the Lie algebra $\mathfrak{g}_3$.

As we have just seen, extremal solutions have $H = 0$. The standard property of 3×3 matrices

$$Q^3 - \mathrm{Tr}(Q)Q^2 + \frac{1}{2}[\mathrm{Tr}(Q^2) - (\mathrm{Tr}Q)^2]Q - \det(Q) = 0 \tag{255}$$

then implies that $Q^3 = 0$, as a matrix equation in the fundamental representation; more intrinsically, in terms of the adjoint representation, this is equivalent to

$$[Ad(Q)]^5 = 0 \tag{256}$$

Thus, Q is a nilpotent element of order 5 in $\mathfrak{g}_3^*$. This condition is invariant under the co-adjoint action of G_3. We conclude that the classical phase space of extremal configurations is a nilpotent coadjoint orbit[21] of G_3. By the general "orbit philosophy" [154], the quantum Hilbert space then furnishes a "unipotent" representation of G_3, obtained by quantizing this nilpotent co-adjoint orbit. As we shall see in Sect. 7.5, this fact extends to the BPS Hilbert space in very special supergravities, where $\mathscr{M}_3^*$ is a symmetric space.

7.3 BPS Black Holes and BPS Geodesics

Up till now, our discussion did not assume any supersymmetry. In general, however, the KK* reduction of the fermions gives extra fermionic contributions in (228), such that the resulting non-linear sigma model has the same amount of supersymmetry as its four-dimensional parent. Moreover, the spherical reduction of the fermions preserves half of the supersymmetries. This leads to the action for a supersymmetric spinning particle moving on $\mathscr{C}$, schematically

$$S_1 = \int d\tau \, \left[g_{ab}\dot\phi^a\dot\phi^b + g_{ab}\, \psi^a D_\tau \psi^b + R_{abcd}\psi^a\psi^b\psi^c\psi^d \right] \tag{257}$$

This Lagrangian is supersymmetric for any target space, but has N-fold extended supersymmetry when $\mathscr{C}$ admits $N-1$ complex structures $J^{(i)}$ $(i = 1,\ldots,N-1)$. The supersymmetry variations of the fermions are then of the form

$$\delta_\epsilon \psi^a = \sum_{i=0}^{N-1} \epsilon^{(i)} J_b^{(i)a} \dot\phi^b + O(\psi^2) \tag{258}$$

with $J_b^{(0)a} = \delta_b^a$ the identity operator. Moreover, the existence of a homothetic Killing vector $r\partial_r$ implies that the action S_1 should be superconformally invariant.

[21] It is a peculiarity of this model that the dimension of this nilpotent orbit is the same – 6 – as that of the generic semi-simple orbits. In general, nilpotent orbits can be much smaller than the generic ones.

BPS solutions in four dimensions correspond to special trajectories on $\mathscr{M}_3^*$, for which there exist a non-zero $\epsilon^{(i)}$ such that the right-hand side of (258) vanishes. This puts a strong constraint on the momentum $p_a = g_{ab}\dot{\phi}^b$ of the fiducial particle, which defines a "BPS" subspace of the phase space $T^*(\mathscr{C})$. The symplectic structure on this BPS phase space can then be obtained using Dirac's theory of Hamiltonian constraints. Due to the existential quantifier $\exists\epsilon^{(i)} \neq 0$, it is sometimes convenient to extend the phase space by including the Killing spinor $\epsilon^{(i)}$, we shall see an example of this in Sect. 7.3.2. In theories with $N \geq 2$ supersymmetry in 4 dimensions, black holes may preserve different fractions of supersymmetry, associated to different orbits of the momentum p under the holonomy group of $\mathscr{C}$. Correspondingly, there will be different BPS phase spaces, nested into each other.

7.3.1 Attractor Flow and BPS Geodesic Flow in $\mathscr{N} = 2$ SUGRA

After this deliberately schematic discussion, we now specialize to $\mathscr{N} = 2$ supergravity, and show that the attractor flow (53),(54) is indeed equivalent to BPS geodesic flow on the three-dimensional moduli space $\mathscr{M}_3^*$.

As explained in Sect. 3, $\mathscr{N} = 2$ supersymmetry determines the metric on $\mathscr{M}_4$ (now denoted $g_{i\bar{j}}$, to take into account the complex nature of the vector multiplet moduli) and gauge couplings $\theta_{IJ} - it_{IJ} \equiv \mathscr{N}_{IJ}$ in terms of a prepotential $F(X)$ via (29), (37). The scalar manifold $\mathscr{M}_3$ obtained by Kaluza-Klein reduction to three dimensions is now a quaternionic-Kähler space, usually referred to as the "c-map" of the special Kähler manifold $\mathscr{M}_4$ [149, 155]. The analytically continued $\mathscr{M}_3^*$, with metric (229) is a pseudo-quaternionic-Kähler space, which we shall refer to as the "c*-map" of $\mathscr{M}_4$. While $\mathscr{M}_3$ has a Riemannian metric with special holonomy $USp(2) \times USp(2n_V + 2)$, $\mathscr{M}_3^*$ has a split signature metric with special holonomy $Sp(2) \times Sp(2n_V + 2)$. For convenience, we will work with the Riemannian space $\mathscr{M}_3$ and perform the analytic continuation at the end.

In order to determine the couplings of the corresponding fermions, one should in principle reduce the four-dimensional fermions along the time direction, then further on the spherically symmetric ansatz (233). For our present purposes, however, it is sufficient to recall that the quaternionic-Kähler space $\mathscr{M}_3$ equivalently arises as the target space of a $N = 2$ supersymmetric sigma model in $3+1$ dimensions, coupled to gravity [156]. Upon reducing the action and supersymmetry transformations of [156] along three flat spatial directions, one obtains a $N = 4$ supersymmetric sigma model in $0+1$ dimensions, which must be identical to the result of the spherical reduction. The supersymmetry variations are then simply

$$\delta_\epsilon \phi^a = O(\psi)\,, \quad \delta_\epsilon \psi^{AA'} = V_a^{AB'}\,\dot{\phi}^a \epsilon_{B'}^{A'} + O(\psi^2) \tag{259}$$

Here, $V^{AA'}$ ($A = 1,..,2n_V + 2$ and $A' = 1,2$) is the "quaternionic viel-bein" afforded by the decomposition

$$T_{\mathbb{C}}\mathscr{M}_3 = E \otimes H \tag{260}$$

of the complexified tangent bundle of $\mathscr{M}_3$, where E and H are complex vector bundles of respective dimensions $2n_V+2$ and 2. Similarly, the Levi-Civita connection decomposes into its $USp(2)$ and $USp(2n_V+2)$ parts p and q,

$$\Omega_{AA'}^{BB'} = p_{B'}^{A'}\epsilon_A^B + q_B^A\epsilon_{A'}^{B'} \tag{261}$$

where $\epsilon_{A'B'}$ and ϵ_{AB} are the antisymmetric tensors invariant under $USp(2)$ and $USp(2n)$. The viel-bein V controls both the metric and the three almost complex structures on the quaternionic-Kähler space,

$$ds^2 = \epsilon_{A'B'}\,\epsilon_{AB}\,V^{AA'}\otimes V^{BB'}, \quad \Omega^i = \epsilon_{A'B'}\,(\sigma^i)_{C'}^{B'}\,\epsilon_{AB}\,V^{AA'}\wedge V^{BC'} \tag{262}$$

(where σ^i, $i=1,2,3$ are the Pauli matrices) and is covariantly constant with respect to the connection (261).

From (259), it is apparent that supersymmetric solutions are obtained when $V^{AA'}$ has a zero right-eigenvector,

$$\text{SUSY} \quad \Leftrightarrow \quad \exists\epsilon_{A'} \neq 0 \;/\; V^{AA'}\epsilon_{A'} = 0 \tag{263a}$$

$$\Leftrightarrow \quad \forall A,B\,, \quad \epsilon_{A'B'}V^{AA'}V^{BB'} = 0 \tag{263b}$$

For fixed $\epsilon^{A'}$, these are $2n_V+2$ conditions on the velocity vector $\dot{\phi}^a$ at any point along the geodesic, removing half of the degrees of freedom from the generic trajectories. In particular, the conditions (263b) imply that

$$\epsilon_{AB}\epsilon_{A'B'}V^{AA'}V^{BB'} = 0 = H\,, \tag{264}$$

and therefore that a BPS solution is automatically extremal. For the universal sector discussed in Sect. 7.2.5, where $n_V=0$, this is actually a necessary and sufficient condition for supersymmetry.

For the case of the c-map $\mathscr{M}_3$, the quaternionic viel-bein was computed explicitly in [149]. After analytic continuation, one obtains

$$V^{AA'} = \begin{pmatrix} iu & v \\ e^a & iE^a \\ -i\bar{E}^{\bar{a}} & \bar{e}^{\bar{a}} \\ -\bar{v} & i\bar{u} \end{pmatrix} \tag{265}$$

where $e^a = e^a_i dz^i$ is a viel-bein of the special Kähler manifold, $e^a_i\bar{e}_{\bar{a}\bar{j}}\delta_{a\bar{a}} = g_{i\bar{j}}$, and

$$u = e^{\mathscr{K}/2-U}X^I\left(d\tilde{\zeta}_I + \mathscr{N}_{IJ}d\zeta^J\right) \tag{266}$$

$$v = -dU + \frac{i}{2}e^{-2U}\left(d\sigma + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I\right) \tag{267}$$

$$E^a = e^{-U}e^a_i g^{i\bar{j}}\bar{f}^I_{\bar{j}}\left(d\tilde{\zeta}_I + \mathscr{N}_{IJ}d\zeta^J\right) \tag{268}$$

Expressing $d\zeta^I$, $d\tilde{\zeta}_I$, $d\sigma$ in terms of the conserved charges (241), the entries in the quaternionic viel-bein may be rewritten as

$$u = -\frac{i}{2} e^{\mathscr{K}/2+U} X^I \left[q_I - 2k\tilde{\zeta}_I - \mathscr{N}_{IJ}(p^J + 2k\zeta^J) \right] d\tau \,, \tag{269}$$

$$v = -dU + \frac{i}{2} e^{2U} k \, d\tau \tag{270}$$

$$e^a = e^a_i \, dz^i \,, \tag{271}$$

$$E^a = -\frac{i}{2} e^U e^{ai} g^{i\bar{j}} \bar{f}^I_{\bar{j}} \left[q_I - 2k\tilde{\zeta}_I - \mathscr{N}_{IJ}(p^J + 2k\zeta^J) \right] d\tau \tag{272}$$

Now, return to the supersymmetry variation of the fermions (259): the existence of $\epsilon^{B'}_{A'}$ such that $\delta\psi^{AA'}$ vanishes implies that the first column of V has to be proportional to the second, hence

$$-\frac{dU}{d\tau} + \frac{i}{2} e^{2U} k = -\frac{i}{2} \, e^{i\theta} \, e^{\mathscr{K}/2+U} \, X^I \left(q_I - k\tilde{\zeta}_I - \mathscr{N}_{IJ}(p^J + k\zeta^J) \right) \tag{273}$$

$$\frac{dz^i}{d\tau} = -\frac{i}{2} \, e^{i\theta} \, e^U \, g^{i\bar{j}} \bar{f}^I_{\bar{j}} \left(q_I - k\tilde{\zeta}_I - \mathscr{N}_{IJ}(p^J + k\zeta^J) \right) \tag{274}$$

where the phase θ is determined by requiring that U stays real. These equations may be rewritten as

$$-\frac{dU}{d\tau} + \frac{i}{2} e^{2U} k = -\frac{i}{2} e^{i\theta} e^U Z \,, \quad \frac{dz^i}{d\tau} = -i e^{i\theta} \frac{|Z|}{Z} e^U g^{i\bar{j}} \partial_{\bar{j}} |Z| \tag{275}$$

where $\hat{Z}$ is the "generalized central charge"

$$\hat{Z}(p,q,k) = e^{\mathscr{K}/2} \left[\hat{q}_I X^I - \hat{p}^I F_I \right] \tag{276}$$

and $\hat{p}^I$, $\hat{q}_I$ have been defined in (246). For vanishing NUT charge, we recognize the attractor flow (53), (54). The equivalence between the attractor flow equations on $\mathscr{M}_4$ and supersymmetric geodesic motion on $\mathscr{M}_3$ was in fact observed long ago in [157] and is a consequence of T-duality between black holes and instantons, after compactifying to three dimensions [158, 159].

This concludes our proof that BPS geodesics, characterized by the BPS constraints (263a), indeed describe stationary, spherically symmetric BPS black holes.

7.3.2 Swann Space and Twistor Space

While the analysis in the previous section identified the BPS subspace of the phase space $T^*\mathscr{M}_3^*$ (namely, the solution to the quadratic constraints (263b)), the non-linearity of the BPS constraints makes it difficult to obtain its precise symplectic structure. We now show that, by lifting the geodesic motion on the quaternionic-Kähler $\mathscr{M}_3^*$ to a higher-dimensional space, namely the Swann space $\mathscr{S}$, it is possible to linearize these constraints.

The Swann space is a standard construction, which relates quaternionic-Kähler geometry in dimension $4n_V+4$ to hyperkähler geometry in $4n_V+8$ dimensions [160]. Namely, let $\pi^{A'}$ $(A'=1,2)$ be complex coordinates in the vector bundle H over $\mathscr{M}_3$, and $\mathscr{S}$ be the total space of this bundle. $\mathscr{S}$ admits a hyperkähler metric

$$ds^2_{\mathscr{S}} = |D\pi|^2 + R^2\, ds^2_{\mathscr{M}_3}\,. \tag{277}$$

where

$$D\pi^{A'} = d\pi^{A'} + p^{A'}_{B'}\pi^{B'}\,, \quad R^2 \equiv |\pi|^2 = |\pi^1|^2 + |\pi^2|^2 \tag{278}$$

In fact, R^2 is the hyperkähler potential of (277), i.e. a Kähler potential for all complex structures. Being hyperkähler, $\mathscr{S}$ has holonomy $USp(2n_V+4)$; the corresponding covariantly constant vielbein $\mathscr{V}^{\aleph}$ (where $\aleph \in \{A, A'\}$ runs over two more indices than A) can be simply obtained from the quaternionic vielbein $V^{AA'}$ on the base $\mathscr{M}_3$ via

$$\mathscr{V}^A = V^{AA'}\pi_{A'}\,, \quad \mathscr{V}^{A'} = D\pi^{A'} \tag{279}$$

The viel-bein $\mathscr{V}^{\aleph}$ gives a set of (1,0)-forms on $\mathscr{S}$ (for a particular complex structure), which together with $\bar{\mathscr{V}}$, span the cotangent space of $\mathscr{S}$. The fermionic variations in the corresponding sigma model split into

$$\delta_\epsilon \psi^{\aleph} = \mathscr{V}^{\aleph}\epsilon + \dots\,, \quad \delta_{\bar\epsilon}\bar\psi^{\bar\aleph} = \bar{\mathscr{V}}^{\aleph}\bar\epsilon + \dots \tag{280}$$

Moreover, the metric (277) has a manifest $SU(2)$ isometry, and homothetic Killing vector $R\partial_R = \pi^{A'}\partial_{\pi^{A'}} + \bar\pi^{A'}\partial_{\bar\pi^{A'}}$. Geodesic motion on $\mathscr{S}$ is therefore equivalent to geodesic motion on the base $\mathscr{M}_3$, provided one restricts to trajectories with zero angular momentum under the $SU(2)$ action (and disregard the motion along the radial direction $R^2=|\pi|^2$). By suitable $SU(2)$ rotation, BPS geodesics on $\mathscr{S}$ can be chosen to be annihilated by δ_ϵ, and so correspond to

$$\forall\, \aleph\,, \qquad \mathscr{V}^{\aleph} = 0 \tag{281}$$

Using (279), this entails

$$V^{AA'}\pi_{A'} = 0\,, \quad D\pi^{A'} = 0 \tag{282}$$

The first condition reproduces the BPS condition (7.45a) on $\mathscr{M}_3$ upon identifying[22] $\pi^{A'}$ with the Killing spinor $\epsilon_{A'}$, while the second can be shown to follow from the Killing spinor conditions in four dimensions, consistently with this identification. The condition (281) shows that BPS trajectories are such that the momentum vector is anti-holomorphic at every point. These BPS constraints are clearly first class, and therefore the extended BPS phase space is the Swann space $\mathscr{S}$ itself, equipped with its Kähler form.

[22] In particular, the radius R of the Swann space $\mathscr{S}$ is equal to the norm of the Killing spinor and must be carefully distinguished from the radius r of the cone $\mathscr{S}$.

While the Swann space has a clear physical motivation, the fiber being identified with the Killing spinor, the fact that one must restrict to $\mathbb{R}^\times \times SU(2)$ invariant trajectories means that it is somewhat too large. In fact, one may perform a symplectic reduction – more precisely, a Kähler quotient – with respect to $U(1) \subset SU(2)$ while keeping most of the pleasant properties of the Swann space. The result, known as the twistor space $\mathscr{Z}$, retains one of the three complex structures of $\mathscr{S}$, which is sufficient for exposing half of the $\mathscr{N}=4$ supersymmetries of (257). To exhibit the structure of $\mathscr{Z}$, it is useful to choose the following coordinates on the unit sphere in $\mathbb{R}^4$,

$$e^{i\varphi} = \sqrt{\pi^2/\bar{\pi}^2}\,, \quad z = \pi^1/\pi^2. \tag{283}$$

where φ is the angular coordinate for the Hopf fibration $U(1) \to S^3 \to S^2$ and z is a stereographic coordinate on $S^2 = \mathbb{CP}^1$. In these coordinates, the metric (277) rewrites as

$$ds^2_{\mathscr{S}} = dR^2 + R^2\left[D\phi^2 + \frac{DzD\bar{z}}{(1+z\bar{z})^2} + ds^2_{\mathscr{M}_3}\right] \tag{284}$$

where

$$Dz \equiv dz - \frac{1}{2}(p_1+ip_2) - 2p_3 z - \frac{1}{2}(p_1 - ip_2)z^2\,, \tag{285}$$
$$D\phi \equiv d\phi + \frac{i}{2(1+z\bar{z})}\left(z[d\bar{z} - (p_1+ip_2)] - \bar{z}[dz - (p_1-ip_2)] - 2ip_3(1-\bar{z}z)\right)$$

and $p_i = \sigma^{A'B'}_{(i)}\, p_{(A'B')}$. The connection term in Dz is sometimes known as the projectivized $USp(2)$ connection. The twistor space $\mathscr{Z}$ is the Kähler quotient of $\mathscr{S}$ by $U(1)$ rotations along ϕ [161]; its metric is therefore given by the last two terms in (284)

$$ds^2_{\mathscr{Z}} = \frac{|Dz|^2}{(1+\bar{z}z)^2} + ds^2_{\mathscr{M}_3}\,. \tag{286}$$

The space $\mathscr{Z}$ is itself an S^2 bundle over $\mathscr{M}_3$ and carries a canonical complex structure, which is an integrable linear combination of the triplet of almost complex structures on $\mathscr{M}_3$. It will also be important that $\mathscr{Z}$ carries a holomorphic contact structure X (proportional to the one-form Dz), inherited from the holomorphic symplectic structure on the hyperkähler cone $\mathscr{S}$.

For later purposes, it will be useful to have an explicit set of $2n_V+3$ complex coordinates $(\xi^I, \tilde{\xi}_I, \alpha)$ on the twistor space $\mathscr{Z}$, adapted to the Heisenberg symmetries, i.e. such that the Killing vectors p^I, q_I, k in (230a) take the standard form

$$p^I = \partial_{\tilde{\xi}_I} - \xi^I\partial_\alpha + \text{c.c.}\,, \quad q_I = -\partial_{\xi^I} - \tilde{\xi}_I\partial_\alpha + \text{c.c.}\,, \quad k = \partial_\alpha + \text{c.c.} \tag{287}$$

while the holomorphic contact structure takes the canonical, Darboux form,

$$X = d\alpha + \tilde{\xi}_I d\xi^I - \xi_I d\tilde{\xi}^I \tag{288}$$

Such a coordinate system has been constructed recently in [11], from which we collect the relevant formulae. The complex coordinates $(\xi^I, \tilde{\xi}_I, \alpha)$ are related to the coordinates $U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I,\ \sigma$ on the quaternionic-Kähler base, as well as the fiber coordinate $z \in \mathbb{C}P_1$, via the "twistor map"

$$\xi^I = \zeta^I + 2i\, e^{U+\mathscr{K}(X,\bar{X})/2} \left(z\bar{X}^I + z^{-1} X^I \right) \tag{289a}$$

$$\tilde{\xi}_I = \tilde{\zeta}_I + 2i\, e^{U+\mathscr{K}(X,\bar{X})/2} \left(z\, \bar{F}_I + z^{-1}\, F_I \right) \tag{289b}$$

$$\alpha = \sigma + \zeta^I \tilde{\xi}_I - \tilde{\zeta}_I \xi^I \tag{289c}$$

These formulae were derived in [11] by using the projective superspace description of the c-map found in [162]. A key feature of these formulae is that, for a fixed point on the base, the complex coordinates $\xi^I, \tilde{\xi}_I, \alpha$ depend rationally on the fiber coordinate $\mathscr{Z}$; said differently, the fiber over any point on the base is rationally in $\mathscr{Z}$. This is a general property of twistor spaces, which allows for the existence of the Penrose transform relating holomorphic functions on $\mathscr{Z}$ to harmonic-type functions on $\mathscr{M}_3$, a topic which we shall return to in Sect. 7.4.3.

The Kähler potential on $\mathscr{Z}$ in these coordinates was also computed in [11] and reads

$$K_{\mathscr{Z}} = \frac{1}{2} \log \left\{ \Sigma^2 \left[\frac{i}{2}(\xi^I - \bar{\xi}^I), \frac{i}{2}(\tilde{\xi}_I - \bar{\tilde{\xi}}_I) \right] + \frac{1}{16} \left[\alpha - \bar{\alpha} + \xi^I \bar{\tilde{\xi}}_I - \bar{\xi}^I \tilde{\xi}_I \right]^2 \right\} + \log 2\,. \tag{290}$$

where $\Sigma_{BH}(\phi^I, \chi_I)$ is the Hesse potential defined in Exercise 8 on page (8). In particular, $K_{\mathscr{Z}}$ is a symplectic invariant, but, as we shall see in Sect. 7.5, it can be invariant under an larger group which mixes $\xi^I, \tilde{\xi}_I$ with α.

The Swann space can be recovered from the twistor space $\mathscr{Z}$ by supplementing the coordinates $\xi^I, \tilde{\xi}_I, \alpha$ with one complex coordinate λ (a coordinate in the $O(-1)$ bundle over $\mathscr{Z}$). The hyperkähler potential on $\mathscr{S}$ and the coordinates $\pi^{A'}$ in the $\mathbb{R}^4$ fiber are then obtained by

$$R^2 = |\lambda|^2\, e^{\mathscr{K}_{\mathscr{Z}}}\,, \qquad \begin{pmatrix} \pi^1 \\ \pi^2 \end{pmatrix} = 2\,\lambda\, e^U \begin{pmatrix} z^{\frac{1}{2}} \\ z^{-\frac{1}{2}} \end{pmatrix}. \tag{291}$$

Using the twistor map (and its converse, which can be found in [11]), it was shown that the holomorphy condition (281) for supersymmetric geodesics on $\mathscr{S}$ allows to fully integrate the motion, reproducing known spherically symmetric black hole solutions.

7.4 Quantum Attractors

We now discuss the radial quantization of stationary, spherically symmetric geometries in four dimensions, using the equivalence between the radial evolution equations and geodesic motion of a fiducial particle on the cone $\mathscr{C} = \mathbb{R}^+ \times \mathscr{M}_3^*$. For brevity, we drop the cone direction and restrict to motion along $\mathscr{M}_3^*$. We start with some generalities in the non-supersymmetric set-up, and then restrict to the BPS sector of $\mathscr{N}=2$ supergravity.

7.4.1 Radial Quantization of Spherically Symmetric Black Holes

Based on the afore-mentionned equivalence, a natural path towards quantization is to replace functions on the classical phase space $T^*(\mathscr{M}_3^*)$ by square integrable functions Φ on $\mathscr{M}_3^*$ and impose the quantum version of the mass-shell condition (237),

$$\left[\Delta_3 + C^2\right]\ \Phi_C(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = 0 \tag{292}$$

Here Δ_3 is the Laplace-Beltrami operator on $\mathscr{M}_3^*$, the quantum analogue of the Hamiltonian $-H$. In writing this, we have ignored the fermionic degrees of freedom, which we shall discuss in the next Sect. 7.4.2, and possible quantum corrections to the energy C^2. In practice, we are interested in wave functions which are eigenmodes of the electric and magnetic charge operators given by the differential operators in (230a),

$$\Phi_C(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = \Phi_{C,p,q}(U, z^i)\ e^{i(p^I \tilde{\zeta}_I - q_I \zeta^I)} \tag{293}$$

which is then automatically a zero eigenmode of the NUT charge k. Note however that, due to the Heisenberg algebra (231a), it is impossible to simultaneously diagonalize the ADM mass operator M, unless either p^I or q_I vanish. Equation (292) then implies that the wave function $\Phi_{C,p,q}(U,\ z^i)$ should satisfy a quantum version of (244),

$$\left[-\partial_U^2 - \Delta_4 - e^{2U} V_{BH}(p,q,z) - C^2\right]\ \Phi_{C,p,q}(U, z^i) = 0 \tag{294}$$

where Δ_4 is now the Laplace-Beltrami on the four-dimensional moduli space $\mathscr{M}_4$. The wave function $\Phi_{C,p,q}(U,\ z^i)$ describes the quantum fluctuations of the scalars z^i as a function of the size e^U of the thermal circle (i.e. effectively as a function of the distance to the horizon). Importantly, the wave function is not uniquely specified by the charges and extremality parameter, as the condition (294) leaves an infinite dimensional Hilbert space; this ambiguity reflects the classical freedom in choosing the values of the 4D moduli at spatial infinity.

An important aspect of any quantization scheme is the definition of the inner product: as in similar instances of mini-superspace quantization, the L_2 norm on the space of functions on $\mathscr{C}$ is inadequate for defining expectation values, since it involves an integration along the "time" direction U at which one is supposed to perform measurements. The customary approach around this problem is to recall the analogy of (294) with the usual Klein-Gordon equation and to replace the L_2 norm on $\mathscr{M}_3^*$ by the Klein-Gordon norm (or Wronskian) at a fixed time U:

$$\langle \Phi | \Phi \rangle = \int dz^i\ d\zeta^I\ d\tilde{\zeta}_I\ d\sigma\ \Phi^*\ \overleftrightarrow{\partial_U}\ \Phi \tag{295}$$

By construction, this is independent of the value of U chosen to evaluate it. A severe drawback of this inner product is that it is not positive definite. This also has a standard remedy in the case of the Klein-Gordon equation, which is to perform a "second quantization" and replace the wave function Φ by an operator; a similar procedure can be followed here, in analogy with "third quantization" in quantum cosmology [163]. This procedure should presumably be relevant for describing multi-centered solutions. Fortunately, for BPS states this problem is void, since, as we shall see in Sect. 7.4.3, the Klein-Gordon product (295) is positive definite when restricted to this sector.

7.4.2 Supersymmetric Quantum Mechanics and BPS Hilbert Space

In the presence of fermionic degrees of freedom, the general discussion in the previous subsection must be slightly amended. Upon quantization, the fermions ψ^a in (257) become Dirac matrices on the target space $\mathscr{M}_3^*$, and the wave function is now valued in $L^2(\mathscr{M}_3^*) \otimes \mathrm{Cl}$, where Cl is the Clifford algebra of $\mathscr{M}_3^*$. Equivalently, one may represent the fermion ψ^a as a differential $d\phi^a$ in the exterior differential algebra on $\mathscr{M}_3^*$ and view the wave function as an element of the de Rham complex of $\mathscr{M}_3^*$, i.e. as a set of differential forms of arbitrary degree [164]. The Wheeler-De Witt (292) now selects eigenmodes of the de Rham Laplacian $d \star d$ with eigenvalue $-C^2$; in particular, for extremal black holes, the wave function becomes an element of the de Rham cohomology of $\mathscr{M}_3^*$. These subtleties do not affect the functional dimension of the Hilbert space, and there still exists a continuum of states with given electric and magnetic charges.

In the presence of extended supersymmetry, however, it becomes possible to look for quantum states which preserve part of the supersymmetries. The simplest example is supersymmetric quantum mechanics on a Kähler manifold [165, 166, 167]: the de Rham complex is refined into the Dolbeault complex, and states annihilated by one-half of the supersymmetries are elements of the Dolbeaut cohomology $H^{p,0}(X)$, isomorphic to the sheaf cohomology group $H^0(X,\Omega^p)$. In more mundane terms, this means that the BPS wave functions are holomorphic differential forms of arbitrary degree, in particular, the functional dimension of the BPS Hilbert space is now $\dim(X)/2$, half the dimension of the Hilbert space for generic ground states.

We now turn to the case of main interest for us, supersymmetric quantum mechanics on a quaternionic-Kähler manifold[23] Classically, we have seen in (7.45a) that supersymmetric solutions are those for which the quaternionic viel-bein $V^{AA'}$ has a zero eigenvector $\epsilon_{A'}$. If we disregard the Killing spinor, the BPS condition is summarized by the quadratic equations in (7.45b). Since $V^{AA'}/d\tau$ is equal to the momentum of the fiducial particle, this is naturally quantized into

$$\forall A,B\ , \qquad \left[\epsilon^{A'B'} \nabla_{AA'} \nabla_{BB'} + \kappa\, \epsilon_{AB}\right] \Phi = 0 \tag{296}$$

[23] This system first appeared in the context of monopole dynamics in $\mathscr{N}=2$ gauge theories [168].

where we allowed for a possible quantum ordering ambiguity κ. Here, $\nabla_{AA'} = V^a_{AA'}\nabla_a$ is the covariant derivative on $\mathscr{M}_3^*$, rotated by the inverse quaternionic viel-bein.

On the other hand, we have seen that it was possible to work in an extended phase space which includes the Killing spinor $\epsilon_{A'}$, and describes geodesic motion on the Swann space $\mathscr{S}$. The supersymmetry condition (281) is now linear in the momentum $V_{\aleph}$ and is naturally quantized into

$$\forall\, \aleph\ , \qquad \bar\partial_{\aleph}\Phi' = 0 \tag{297}$$

where $\bar\partial_{\aleph}$ are partial derivatives with respect to a set of antiholomorphic coordinates $\bar z^{\bar\aleph}$ on $\mathscr{S}$. Thus, wave functions on the extended phase space are just holomorphic functions on $\mathscr{S}$ (or more accurately, elements of the sheaf cohomology of $\mathscr{S}$).

Since the classical geodesic motions on $\mathscr{M}_3^*$ and $\mathscr{S}$ are equivalent only for trajectories with vanishing $SU(2)$ momentum, it should be possible to generate a solution of the second-order differential equation (296) from a holomorphic function on $\mathscr{S}$, by projecting on $\mathbb{R}^\times \times SU(2)$ invariant states. Part of this projection can already be taken care of by restricting to homogeneous functions of fixed degree $-k$ on $\mathscr{S}$, or equivalently to sections of $O(-k)$ on $\mathscr{Z}$.

7.4.3 Quaternionic Penrose Transform and Exact BPS Wave Function

Remarkably, there does exist a mathematical construction valid for any quaternionic-Kähler manifold, sometimes known as the quaternionic Penrose transform [11, 169, 170], which performs exactly this task, namely takes an element of $H^1(\mathscr{Z},O(-2))$ to a solution of (296). This is an analogue of the more familiar Penrose transform which maps sections of $H^1(\mathbb{CP}^3,O(-2)$ to massless spin 0 fields on $\mathbb{R}^4$ [171]. Using the complex coordinate system introduced in Sect. 7.3.2, it is easy to provide an explicit integral representation of this transform, where the element of $H^1(\mathscr{Z},O(-2))$ is represented by a holomorphic function $g(\xi^I,\tilde\xi_I,\alpha)$ in the trivialization $\lambda = 1$ [11]:

$$\Phi\left(U,z^a,\bar z^{\bar a},\zeta^I,\tilde\zeta_I,\sigma\right) = e^{2U}\oint \frac{dz}{z}\, g\left(\xi^I,\tilde\xi_I,\alpha\right), \tag{298}$$

In this formula, $\xi^I,\tilde\xi_I,\alpha$ are to be expressed as functions of the coordinates on $\mathscr{M}_3$ and z via the twistor map (289a). The integral runs over a closed contour which separates $z=0$ from $z=\infty$. In [11], it was shown that the left-hand side of (298) is indeed a solution of the system of second-order differential equations (296) with with a fixed value for $\kappa = -1$. Moreover, the Klein-Gordon inner product on $\mathscr{M}_3$ (295) may be rewritten in terms of the holomorphic function g as

$$\langle\Phi|\Phi'\rangle = \int d\xi^I d\tilde\xi_I d\alpha\, d\bar\xi^I d\bar{\tilde\xi}_I d\bar\alpha\; e^{-2(n_V+1)K_{\mathscr{Z}}}\, \overline{g\left(\xi^I,\tilde\xi_I,\alpha\right)}\, g'\left(\xi^I,\tilde\xi_I,\alpha\right) \tag{299}$$

where the integral runs over values of $\xi^I,\ \tilde{\xi}_I,\ \alpha,\ \bar{\xi}^I,\ \bar{\tilde{\xi}}_I,\ \bar{\alpha}$ such that the bracket in (290) is strictly positive. In particular, the inner product (299) is positive definite, as announced at the end of Sect. 7.4.1.

There also exist versions of (298), (299) appropriate to sections of $H^1(\mathscr{Z}, O(-k))$ for any $k > 0$, which are mapped to sections of $\Lambda^{k-2}(H)$ satisfying first-order differential equations [11].

Thus the problem of determining the radial wave function of BPS black holes is reduced to that of finding the appropriate section of $H^1(\mathscr{Z},\ O(-2))$. For a black hole with fixed electric and magnetic charges $q_I,\ p_I$ and zero NUT charge, the only eigenmode of the generators (287) is, up to normalization, the "coherent state"

$$g_{p,q}(\xi^I, \tilde{\xi}_I, \alpha) = e^{\mathrm{i}(p^I \tilde{\xi}_I - q_I \xi^I)} . \tag{300}$$

These states are delta-normalizable under inner product (299) (possibly regulated by analytic continuation in k) and become normalizable after modding out by the discrete Heisenberg group[24].

Applying the Penrose transform (298) to the state (300), we find

$$\Phi_{p,q}(U, z^a, \bar{z}^{\bar{a}}, \zeta^I, \tilde{\zeta}_I, \sigma) = e^{\mathrm{i}p^I \tilde{\zeta}_I - \mathrm{i}q_I \zeta^I}\, e^{2U} \oint \frac{dz}{z} \exp\left[e^U (z\bar{Z} + z^{-1} Z)\right], \tag{301}$$

where Z is the central charge (41) of the black hole. After analytic continuation $(\zeta^I, \tilde{\zeta}_I)$ to $i(\zeta^I, \tilde{\zeta}_I)$ and (p^I, q_I) to $-i(p^I, q_I)$, the integral may be evaluated in terms of a Bessel function,

$$\Phi(U, z^a, \bar{z}^{\bar{a}}, \zeta^I, \tilde{\zeta}_I, \sigma) = 2\pi\, e^{\mathrm{i}p^I \tilde{\zeta}_I - \mathrm{i}q_I \zeta^I}\, e^{2U}\, J_0\left(2e^U |Z|\right) \tag{302}$$

This is the exact radial wave function for a black hole with fixed charges (p^I, q_I), at least in the supergravity approximation[25].

Since the Bessel function J_0 decays like $\cos(w)/\sqrt{w}$ at large values of $|w|$, we see that the phase of the BPS black hole wave function is stationary at the classical attractor point $z^i_{p,q}$, and becomes flatter and flatter in the near-horizon limit $U \to -\infty$, while the modulus decays away from these points as a power law. The occurrence of large quantum fluctuations in the near horizon limit may seem at odds with the attractor behavior for BPS black holes, but is in fact perfectly consistent with the picture of a particle moving in an inverted potential $V = -e^{2U} V_{BH}$, as discussed in Sect. 7.2.4. It is a reflection of the infinite fine-tuning of the asymptotic conditions which is necessary for obtaining an extremal black hole.

Returning to the original motivation explained in Sect. 7.1, we observe that the wave function (302) bears no obvious relation to the topological string amplitude. One may, however, try to rescue the suggestion in [7] by noting that there is in principle an even smaller subspace of the Hilbert space $L^2(\mathscr{S})$, corresponding to "tri-holomorphic" on $\mathscr{S}$; we shall remain deliberately vague about the concept of

[24] Scaling arguments show that the norm grows as a power of p, q, rather than exponentially.

[25] In the presence of R^2-type corrections, the geodesic motion receives higher-derivative corrections, and it is no longer clear how to quantize it.

"tri-holomorphy" here, referring the reader to [172] for some background on this subject, but merely assume that it divides the functional dimension by a factor of four. If so, this "super-BPS" Hilbert space of triholomorphic functions on $\mathscr{S}$ would have functional dimension n_V+2, and be the natural habitat of a one-parameter generalization of the topological wave function [9]. One would also expect some quaternionic analogue of the Cauchy integral in (298), which would map the space of tri-holomorphic functions on $\mathscr{S}$ to functions on $\mathscr{M}_3$ annihilated by certain differential operators. In the symmetric cases studied in the next Section, we shall indeed be able to construct a "super-BPS" Hilbert space, of functional dimension n_V+2, which carries the smallest possible unitary representation of the duality group.

7.5 Very Special Quantum Attractors

We now specialize the construction of Sect. 7.4.2 to the case of very special $\mathscr{N}=2$ supergravities which we introduced in Sect. 3.5. Our goal is to produce a framework for constructing duality-invariant black hole partition functions, applicable both for these $\mathscr{N}=2$ theories and their $\mathscr{N}=4,8$ variants.

7.5.1 Quasiconformal Action and Twistors

Recall that the vector-multiplet moduli space of very special supergravities are hermitean symmetric tube domains (86), built out of the invariance groups of Jordan algebras J with a cubic norm N. The result of the c-map [173] and c^* map [147] constructions are still symmetric spaces, of the form

$$\mathscr{M}_3 = \frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J)\times SU(2)}\,, \quad \mathscr{M}_3^* = \frac{\mathrm{QConf}(J)}{\mathrm{Conf}(J)\times Sl(2)} \tag{303}$$

Here, $\mathrm{QConf}(J)$ is the "quasi-conformal group" associated to the Jordan algebra J (in its quaternionic, rank 4 real form), and $\widetilde{\mathrm{Conf}}(J)$ is the compact real form of $\mathrm{Conf}(J)$; these spaces can read off from Table 2 on page 26.

The terminology of "quasi-conformal group" refers to the realization found in [174] of $G=\mathrm{QConf}(J)$ as the invariance group of the zero locus $\mathscr{N}_4(\Xi,\bar\Xi)=0$ of a homogeneous, degree 4 polynomial $\mathscr{N}_4$ in the variables $\Xi=(\xi^I,\tilde\xi_I,\alpha)$ (of respective degree 1,1,2) and $\bar\Xi=(\bar\xi^I,\bar{\tilde\xi}_I,\bar\alpha)$:

$$\mathscr{N}_4(\Xi;\bar\Xi)=\frac{1}{2}I_4\left(\xi^I-\bar\xi^I,\tilde\xi_I-\bar{\tilde\xi}_I\right)+\left(\alpha-\bar\alpha+\bar\xi^I\tilde\xi_I-\xi^I\bar{\tilde\xi}_I\right)^2 \tag{304}$$

More precisely, there exists an holomorphic action of G on Ξ such that $\mathscr{N}_4(\Xi,\bar\Xi)$ gets multiplied by a product $f(\Xi)\bar f(\bar\Xi)$. In (304), I_4 is the quartic invariant (96) of the group $\mathrm{Conf}(J)\subset\mathrm{QConf}(J)$ associated to the Jordan algebra J, acting linearly on the symplectic vectors $(\xi^I,\tilde\xi_I)$ and $(\bar\xi^I,\bar{\tilde\xi}_I)$. By analogy with the "conformal realization" of Conf(J), leaving the cubic light-cone $N(z^i - \bar{z}^i)$ invariant, this is called the quasi-conformal realization of G.

Group theoretically, the origin of this action is clear: the group G admits a 5-graded decomposition, corresponding to the horizontal axis in the two-dimensional projection of the root diagram of G shown in Fig. 6),

$$\begin{aligned} G = G_{-2} \oplus \quad G_{-1} \quad \oplus \qquad G_0 \qquad \oplus \quad G_{+1} \quad \oplus G_{+2} \\ \equiv \{k'\} \oplus \{p'_I, q'^I\} \oplus \{T_A, S^A, D_A^B\} \oplus \{p^I, q_I\} \oplus \{k\} \end{aligned} \tag{305}$$

In particular, the top space G_{+2} is one-dimensional, therefore $G_{+1} \oplus G_{+2}$ form an Heisenberg algebra with center G_{+2}, which we identify with the Heisenberg algebra $[p^I, q_J] = 2k\delta_J^I$ of electric, magnetic, and NUT isometries (242). The grade 0 space is $G_0 = \mathrm{Conf}(J) \times U(1) = \{T_A, S^A, D_A^B\}$. Symmetrically, $G_{-1} \oplus G_{-2}$ form an Heisenberg algebra $[p'_I, q'^J] = 2k'\delta_J^I$ with one-dimensional center G_{-2}. Together with $G_{-2} = \{k'\}$ and $G_{+2} = \{k\}$, the center $H = D_A^A = [k, k']$ of G_0 generates an $SU(2)$ subgroup which commutes with Conf(J), and yields the above 5-grading above. Finally, Conf(J) acts linearly on $G_{+1} \sim \{p^I, q_I\}$ in the usual way, leaving $G_{+2} \sim \{k\}$ invariant. Since the H charge is additive, the sum $P = G_{-2} \oplus G_{-1} \oplus G_0$ closes under commutation, and is known as the Heisenberg parabolic subgroup P of G. The quasi-conformal realization of G is then just the action on $P\backslash G$ by right multiplication; it may be twisted by a unitary character χ of P, i.e. by considering functions on $P\backslash G$ which transform by χ under the right action of G (mathematically, this is the induced representation from the parabolic P to G with character χ, see e.g. [175] for an introduction to this concept).

To be completely explicit, the generators in $G_{+1} \oplus G_{+2}$ act on functions of Ξ as

$$E_{p^I} = \partial_{\tilde{\xi}_I} - \xi^I \partial_\alpha \ , \quad E_{q_I} = -\partial_{\xi^I} - \tilde{\xi}_I \partial_\alpha \ , \quad E_k = \partial_\alpha \ , \tag{306}$$

while the generator k' in G_{-2} acts as

$$E_{k'} = \left(-\frac{1}{4} \frac{\partial I_4}{\partial \tilde{\xi}_I} - \alpha \xi^I \right) \partial_{\xi^I} + \left(\frac{1}{4} \frac{\partial I_4}{\partial \xi^I} - \alpha \tilde{\xi}_I \right) \partial_{\tilde{\xi}_I} + \frac{1}{2}(I_4 - 2\alpha^2)\partial_\alpha - k\alpha \tag{307}$$

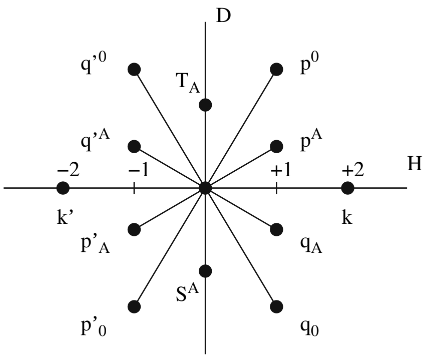

Fig. 6 Two-dimensional projection of the root diagram of the quasi-conformal group associated to a cubic Jordan algebra J (when $J = \mathbb{R}$, this is the root diagram of G_2). The five-grading corresponds to the horizontal axis. The long roots are singlets, generating a $SU(2,\ 1)$ universal subgroup, while the short roots are valued in the Jordan algebra J

where $I_4 = I_4(\xi^I, \tilde{\xi}_I)$ and k is a complex number parametrizing the character χ. The rest of the generators can be obtained by commutation and $\mathrm{Conf}(J)$ rotations.

Comparing (304) and (290) and recalling that the Hesse potential Σ for symmetric spaces is the square root of the quartic invariant I_4, it is manifest that the log of the "quartic light-cone" (304) is just the Kähler potential (290) of the twistor space $\mathscr{Z} = G_{\mathbb{C}} \backslash P_{\mathbb{C}}$ of the quaternionic-Kähler space $\mathscr{M}_3$; therefore, the quasi-conformal realization is nothing but the holomorphic action of $\mathrm{QConf}(J)$ on the twistor space $\mathscr{Z}$. For integer values of the parameter k, this representation belongs to the "quaternionic discrete series" representation of G [176], a quaternionic analogue of the usual discrete series for $Sl(2)$.

We conclude that for very special supergravities, the BPS Hilbert space carries a unitary representation of the three-dimensional U-duality group $G = \mathrm{QConf}(J)$, given by the "quaternionic discrete series" or "quasi-conformal realization" of G.

7.5.2 Penrose Transform and Spherical Vector

In Sect. 7.4.3, we have seen that there is a Penrose transform which takes holomorphic functions on $\mathscr{Z}$ to a function on $\mathscr{M}_3$ annihilated by some second-order differential operator. In the present symmetric context, there is an a priori different way of producing a function Φ on G/K from a vector $f \in \mathscr{H}$ in a unitary representation of G: for any $e \in G$, take

$$\Phi(e) = \langle f | \rho(e) | f_K \rangle \tag{308}$$

where f_K is a fixed K-invariant vector in $\mathscr{H}$. Since f_K is invariant under K, Φ descends to the quotient G/K. This construction is standard in representation theory, where f_K is referred to as a spherical vector (see again [175]).

Not surprisingly, the geometric and algebraic constructions are in fact equivalent, as we illustrate in the simplest case of the universal sector $G = SU(2,1)/SU(2) \times U(1)$. The quartic invariant in this case is the square of a quadric, $I_4 = \frac{1}{2}(\xi^2 + \tilde{\xi}^2)^2$. Using (307) one may check that

$$f_K(\xi, \tilde{\xi}, \alpha) = \left(1 + \xi^2 + \tilde{\xi}^2 + \alpha^2 + \frac{1}{2} I_4\right)^{-k/2} \tag{309}$$

is the unique vector invariant under $SU(2) \times U(1)$. Acting with $\rho(e)$ where $e \in G$ is parameterized by $U, \zeta, \tilde{\zeta}, \sigma$, one obtains

$$[\rho(e) f_K](\xi, \tilde{\xi}, \alpha) = \left[e^{2U} + (\tilde{\xi} - \tilde{\zeta})^2 + (\xi - \zeta)^2 + e^{-2U} \mathscr{N}_4(\xi, \tilde{\xi}, \alpha; \zeta, \tilde{\zeta}, \sigma)\right]^{-k/2} \tag{310}$$

Thus, we conclude that BPS wave functions, in the unconstrained Hilbert space $\mathscr{H}$, are given by

$$\Phi(U,\zeta,\tilde{\zeta},\sigma) = \int \frac{\bar{f}(\bar{\xi},\bar{\tilde{\xi}},\bar{\alpha})\, d\bar{\xi}\, d\bar{\tilde{\xi}}\, d\bar{\alpha}}{\left[e^{2U} + (\bar{\tilde{\xi}} - \tilde{\zeta})^2 + (\bar{\xi} - \zeta)^2 + e^{-2U}\mathcal{N}_4(\zeta,\tilde{\zeta},\sigma,\bar{\xi},\bar{\tilde{\xi}},\bar{\alpha};)\right]^{k/2}} \tag{311}$$

By evaluating the contour integral by residues, one may easily show that, for $k = 2$, the function Φ in (308) agrees with the Penrose transform (298) of the function

$$g(\xi,\tilde{\xi},\alpha) = \int \frac{\bar{f}(\bar{\xi},\bar{\tilde{\xi}},\bar{\alpha})\, d\bar{\xi} d\bar{\tilde{\xi}} d\bar{\alpha}}{(\alpha - \bar{\alpha} + \bar{\tilde{\xi}}\xi - \tilde{\xi}\bar{\xi})^2 + \frac{1}{4}\left[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2\right]^2} \tag{312}$$

This operator which intertwines between the space of functions $g(\xi,\ \tilde{\xi},\ \alpha)$ and $\bar{f}(\bar{\xi},\ \bar{\tilde{\xi}},\ \bar{\alpha})$ is an example of the "twistor transform" (not to be confused with the Penrose transform), which maps sections of $H^1(\mathscr{Z}, O(-k))$ to sections of $H^1(\mathscr{Z}, O(-4-k)$ [177].

7.5.3 The Minimal Representation vs. the Topological Amplitude

At the end of Sect. 7.4.3, we pointed out that the functional dimension of the BPS Hilbert space $H_1(\mathscr{Z}, O(-2))$, $2n_V+3$, was too large to accommodate the topological string amplitude, which depends on n_V+1 variables. In the symmetric case, it is natural to ask whether there are smaller representations than the quasi-conformal realization, which could provide the natural habitat for the topological string amplitude.

In fact, it is known in the mathematics literature that the quasiconformal representation, for low values of the parameter k, is no longer irreducible [176]. In particular, the symplectic space $V = \{\xi^I,\ \tilde{\xi}_I\}$ admits a sequence of subspaces $V \supset X \supset Y \supset Z$, defined by homogeneous polynomial equations of degree 4,3 and 2, respectively such that each of them is preserved by the quasi-conformal action of G. Here, X is the locus where the quartic invariant $I_4(\xi,\tilde{\xi})$ vanishes, $Y \subset X$ is the locus where the differential dI_4 vanishes; finally, $Z \subset Y$ is the locus where the irreducible component of the Hessian of I_4 (viewed as an element of the symmetric tensor product $V \otimes_S V$) transforming in the adjoint representation of $\mathrm{Conf}(J)$ vanishes, a condition which we'll denote $d^2\ I_4 = 0$. As shown in [176], each of the subspaces X, Y, Z, supplemented with the variable α and for the appropriate choice of k, furnishes an irreducible unitary representation of G, of functional dimension $2n_V+3$, $2n_V+2$, $(5n_V+3)/3$ and n_V+2 variables, respectively. By the "orbit philosophy", these are associated by to nilpotent co-adjoint orbits of nilpotency order 5,4,3,2, respectively.

The smallest of those, known as the minimal representation of G, is of particular importance to us, as its dimension n_V+2 is just one more than the number of variables appearing in the topological amplitude. This representation plays a distinguished role in mathematics, being the smallest unitary representation of G and an analogue of the metaplectic representation of the symplectic group.

In physics, the minimal representation of $Sl(3)$ was used in the early days for strong interactions [178], and more recently in an attempt at quantizing BPS membranes [179, 180]. Its relevance to black hole physics was suggested in [181] and expounded in [8, 10].

As first observed in the case of $E_{8(8)}$ in [181], the minimal representation can be obtained by quantizing the symplectic space[26] V of the quasi-conformal realization was acting, namely replacing $\tilde{\xi}_I \to i\partial_{\xi^I}$ and fixing the ordering ambiguities so that the algebra of QConf(J) is preserved. An independent construction, valid for all simply laced cases, was given in [182, 183]; a recent unified approach using the language of Jordan algebra and Freudenthal triple systems can be found in [184, 185].

In order to extract physical information from wave functions in the minimal representation, just as in the quasiconformal case it is necessary to embed them in the non-BPS Hilbert space, i.e. map them into functions on $\mathscr{M}_3$ by some analogue of the Penrose transform. As explained in the previous subsection, this may be done once a spherical vector f_K is found. A slight complication is that the minimal representation for non-compact groups in the quaternionic real form (as opposed to the split real form) do not admit a spherical vector; rather, the decomposition of the minimal representation under the maximal compact group $K = \widetilde{\text{Conf}}(J) \times SU(2)$ has a "ladder" structure, whose lowest component (or "lowest K-type") transforms in a spin[27] $(n_V - 3)/6$ representation of $SU(2)$. Replacing f_K in (308) by this lowest K-type, one obtains a section of a symmetric power of H on $\mathscr{M}_3$. The wave function of the lowest K-type can be computed explicitely in a mixed real-holomorphic polarization [13]; in the semi-classical approximations, all components of the K-type behave as

$$f_K(a^A, b^\dagger, x) \sim \exp\left[-\frac{x^2}{2} + \frac{I_3(a^A)}{b^\dagger} + 2ix\sqrt{\frac{I_3(a^A)}{b^\dagger}}\right] \tag{313}$$

where $f_K(a^A, b^\dagger, x)$ is related to $f_K(\xi^0, \xi^A, \alpha)$ by a certain Bogolioubov operator [13]. We take the fact that f_K reduces to the classical topological amplitude $\exp(I_3(a^A)/b^\dagger)$ in the limit $x \to 0$ as a strong indication that the minimal representation is the habitat of a one-parameter generalization of the standard topological amplitude.

Further evidence for this claim comes from the fact the holomorphic anomaly (148) obeyed by the usual topological amplitude follow from the quadratic identities in the universal enveloping algebra of the minimal representation of G, upon restriction to the "Fourier-Jacobi group" $P/U(1)$, where $U(1)$ is the subgroup generated by the Cartan generator $H \in G_0$ [9]. This is in precise analogy with the heat equation satisfied by the classical Jacobi theta series,

$$\left[i\partial_\tau - \partial_z^2\right] \theta_1(\tau, z) = 0 \tag{314}$$

[26] This is sometimes referred to as "quantizing the quasi-conformal action", which may cause some confusion since the quasi-conformal realization is quantum mechanical already.

[27] For $n_V < 3$, the lowest K-type is a singlet of $SU(2)$ but non-singlet of Conf(J).

which follows from quadratic relations in the minimal (i.e. metaplectic) representation of $Sp(4) \supset Sl(2)$. In this restriction, the generator $k \in G_{+2}$ becomes central and can be fixed to an arbitrary non-zero value, reducing the total number of variables from $n_V + 2$ down to $n_V + 1$. The usual topological amplitude $\Psi_{\mathbb{R}}(p^I)$ should then arise as a "Fourier-Jacobi" coefficient of a "generalized topological amplitude" $\Psi_{\rm gen}(p^I,\, k)$ at $k = 1$. The extension of these considerations to realistic cases without symmetry, possibly along the lines explained at the end of Sect. 7.4.3, would clearly have far-reaching consequences for the enumerative geometry of Calabi-Yau spaces.

7.6 Automorphic Partition Functions

We now return to our original motivation for investigating the radial quantization of BPS black holes, namely, the construction of partition functions for black hole micro-states consistent with the symmetries of the problem. We shall mainly consider the toy model case of very special $\mathcal{N} = 2$ supergravities but will briefly discuss the applications to $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity at the end of this section.

In the previous sections, we discussed how the mini-superspace radial quantization of BPS black holes gives rise to Hilbert spaces of finite functional dimension, furnishing a unitary representation of the three-dimensional duality group $G_3 = \mathrm{QConf}(J)$. It is natural to expect that G_3 should serve as a spectrum-generating symmetry for black hole micro-states [8, 174, 186, 187]. Indeed, it already serves as solution-generating symmetry at the classical level, although it mixes bona-fide black holes with solutions with non-zero NUT charge k. Thus, we propose that the black hole indexed degeneracies $\Omega(p,\, q)$ be given by Fourier coefficients of an automorphic form Z on the three-dimensional moduli space $\mathcal{M}_3 = G(\mathbb{Z})\backslash G/K$. More specifically, consider

$$\Omega(p^I, q_I; U, z^i, \bar{z}^{\bar{i}}) = \int d\zeta^I \, d\tilde{\zeta}_I \, d\sigma \; e^{-ip^I \tilde{\zeta}_I + iq_I \zeta^I} \, Z(U, z^i, \bar{z}^{\bar{i}}; \zeta^I, \tilde{\zeta}_I, \sigma) \tag{315}$$

where the integral runs over a fundamental domain $0 \le (\zeta^I, \tilde{\zeta}_I, \sigma) \le 2\pi$ of the discrete Heisenberg group. The left-hand side is in principle a function of $U,\ z^i, \bar{z}^{\bar{i}}$: one should view Z as the partition function in a thermodynamical ensemble with electric and magnetic potentials ζ^I and $\tilde{\zeta}_I$, temperature $T = e^{-U} m_P$ and values $(z^i, \bar{z}^{\bar{i}})$ for the vector-multiplet moduli at infinity. Provided Z is annihilated by appropriate differential operators, the dependence on $U, z^i, \bar{z}^{\bar{i}}$ will be entirely fixed by the charges p^I, q_I, and leave an overall factor identified as the actual black hole degeneracy:

$$\Omega(p^I, q_I; U, z^i, \bar{z}^{\bar{i}}) = \Omega(p^I, q_I) \; \Phi_{p,q}(U, z^i, \bar{z}^{\bar{i}}) \tag{316}$$

Now, there is a natural way to construct an automorphic form which satisfies these requirements: for $e \in G$, take

$$Z(e) = \langle f_{\mathbb{Z}} | \rho(e) | f_K \rangle \tag{317}$$

where ρ is a unitary representation of G, K a spherical vector and $f_{\mathbb{Z}}$ a $G(\mathbb{Z})$-invariant vector in this representation. This last condition guarantees that $Z(g)$ so defined is a function on $G(\mathbb{Z})\backslash G/H$. We comment on ways to compute $f_{\mathbb{Z}}$ below. In particular, one may take for ρ the quasi-conformal representation described in Sect. 7.5.1: the function $\Phi_{p,q}$ in (316) is then just the black hole wave function (302) (with the dependence on ζ^I and $\tilde{\zeta}_I$ stripped off), while the integer degeneracies $\Omega(p,\ q)$ are encoded in the $G(\mathbb{Z})$-invariant vector $f_{\mathbb{Z}}$. In this case, it is known that the Fourier coefficients have support only on charges with $I_4(p^I, q_I) \geq 0$ [188]. One could also consider smaller representations associated to the subspaces X, Y or Z of V: the coefficients $\Omega(p,\ q)$ would then have support on charges with $I_4(p,\ q) = 0$, $dI_4 = 0$ or $d^2 I_4 = 0$, and would presumably be relevant for "small" black holes with 3, 2 and 1 charges, respectively.

Thus, we have reduced the problem of computing the black hole partition function to that of constructing a $G(\mathbb{Z})$-invariant vector in a unitary representation ρ of the three-dimensional duality group $G(\mathbb{Z})$ [8]. This is a difficult problem, but there is a powerful mathematical method, known as the Strong Approximation Theorem, which allows to address this question (see [175] for a pedestrian introduction to these techniques): this theorem states that functions on $G(\mathbb{Z})\backslash G(\mathbb{R})$ are equivalent to functions on $G(\mathbb{A})/G(\mathbb{Q})$, where $\mathbb{A}$ is the field of adeles, i.e. the (restricted) product of $\mathbb{R}$ times the p-adic number fields $\mathbb{Q}_p$ for all prime p, with $\mathbb{Q}$ being diagonally embedded in this product. Since $G(\mathbb{Q})$ is the maximal compact subgroup of $G(\mathbb{A})$, the problem of finding $f_{\mathbb{Z}}$ is reduced to that of finding the spherical vector over each p-adic field. This point of view has been applied to find the $G(\mathbb{Z})$-invariant vector of the minimal representation for simply laced groups in the real form in [189]. It would be very interesting to construct the automorphic forms attached to quasi-conformal representation and see whether their Fourier coefficients have the required exponential growth.

We close this section by noting that the construction of automorphic partition functions outlined in this section can also be applied, after suitable analytic continuation, to the case of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity, which have a clear string theory realization. While the three-dimensional moduli space is no longer quaternionic-Kähler, there are still unitary representations associated to the symplectic space V and its subspaces X, Y, Z, and one can still define Fourier coefficients of the type (315). For $\mathcal{N} = 8$ supergravity, we expect that exact degeneracies of 1/8-BPS, 1/4-BPS and 1/2-BPS black holes to be given by automorphic forms of $E_{8(8)}$ based on V, Y, Z, respectively (since the 1/4 and 1/2 BPS conditions are $dI_4(p,\ q) = 0$ and $d^2 I_4(p,\ q) = 0$, respectively [186]). For $\mathcal{N} = 4$ supergravity, we expect 1/4-BPS states to be counted by an automorphic form of $SO(8,\ n_v + 2)$ (where n_v is the number of $\mathcal{N} = 4$ vector multiplets in 4 dimensions). This proposal is distinct from the genus 2 partition function outlined in Sect. 2.5, and would have to be consistent with it at least in the large charge regime. In this respect, it is interesting to remark (see Exercise 21 below) that $Sp(4)$ can be viewed as a "degeneration" of the three-dimensional U-duality group QConf(J) (for any J), upon collapsing all electric and magnetic charges p^I and q_I to just two charges p, q. Thus, our proposal has the potential to resolve differences between black holes which have the same continuous U-duality invariant but sit in different orbits of the discrete U-duality group.

Exercise 21. *Show that the root diagram of $Sp(4)$ is "tic-tac-toe"-shaped. Compare to the root diagram of $QConf(J)$ in Fig. 6 on page* 359.

8 Conclusion

In these lectures, we have reviewed some recent attempts at generalizing the microscopic counting of BPS black holes beyond leading order. Our main emphasis was on the conjecture by Ooguri, Strominger, and Vafa, which relates the microscopic degeneracies of four-dimensional BPS black holes to the topological string amplitude, which captures an infinite series of higher derivative corrections in the macroscopic, low energy theory.

By analyzing the case of "small" black holes, which can be easily counted in the heterotic description, we have found that the topological amplitude captures the microscopic degeneracies with impressive precision. At the same time, it is clear that some kind of non-perturbative generalization of the topological string is required, if one wants to obtain exact agreement for finite charges.

Motivated by the "holographic" interpretation of the OSV conjecture as a channel duality between radial and time-like quantization, we studied the quantization of the attractor flow for stationary, spherically symmetric black holes; this was achieved by reformulating the attractor flow as a BPS geodesic flow on the moduli space in three dimensions. Using the Penrose transform, we were able to compute the exact radial wave function for BPS black holes with fixed electric and magnetic charges, in the supergravity approximation. It would be interesting to try and include the effect of higher derivative corrections, as well as relax the assumption of spherical symmetry.

Contrary to the suggestion in [7], the BPS wave function bears little ressemblance to the topological string amplitude. There is, however, evidence from the symmetric space case that there exists a "super-BPS" Hilbert space which can host the topological string wave function or rather a one-parameter generalization thereof. In the general non-symmetric case, this generalized topological amplitude should be viewed as a tri-holomorphic function over the quaternionic-Kähler moduli space (or rather, the Swann space thereof). Using T-duality between the vector-multiplet and hyper-multiplet branches in 3 dimensions, it is natural to expect that it should encode instanton corrections to the hypermultiplet geometry in 4 dimensions [190].

These considerations lend support to the idea that the three-dimensional duality group should play a role as a spectrum-generating symmetry for 4-dimensional black holes. Our framework suggests that the black hole degeneracies should be indeed be related to Fourier coefficients of automorphic forms for the three-dimensional U-duality group G, attached to the representations of G which appear in the radial quantization of stationary, spherically symmetric BPS black holes. It would be interesting to construct these automorphic forms explicitly and have a handle on the growth of their Fourier coefficients, similar to the Rademacher formula for modular forms of $Sl(2,\mathbb{Z})$.

The most direct application of our framework is to BPS black holes in the FHSV model, since this is a quantum realization of the very special $\mathcal{N}=2$ supergravity with $J=\mathbb{R}\oplus\Gamma_{9,1}$; in this case, we expect that the black hole partition function is an automorphic form of $SO(4,\ 12,\ \mathbb{Z})$, which would be very interesting to construct. With some minor amendments, our framework also applies to $\mathcal{N}=4$ and $\mathcal{N}=8$ backgrounds in string theory, whose three-dimensional U-duality groups are $SO(8,\ 24,\ \mathbb{Z})$ and $E_{8(8)}(\mathbb{Z})$. In the $\mathcal{N}=4$ case, our proposal differs from the DVV formula, which relies on an automorphic form of $Sp(4,\ \mathbb{Z})$ but has the potential to distinguish black holes which have the same continuous U-duality invariant but sit in different orbits of the discrete U-duality group. For $\mathcal{N}=8$, the entropy of 1/8-BPS BPS black holes in a certain orbit was computed using the 4D/5D lift in [8, 191]. It would be interesting to see whether an agreement with these formulae can be reached at least for certain orbits.

The extension of these ideas to general $\mathcal{N}=2$ string theories, possibly using the monodromy group of X as a replacement for the U-duality group, is of course the most challenging and potentially rewarding problem, as it is bound to unravel new relations between number theory, algebraic geometry, and physics.

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